Competition Creates Competition: Stock Prices and the Discovery of Takeover Bidders

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Abstract

A stronger incumbent bidder can induce rival entry through a more informative stock price. I study a takeover auction in which a potential rival bidder sees the target's stock price before it pays to prepare a bid. When the high class dominates the low class, stronger incumbents never raise acquisition profit at any belief or narrow the target-payoff spread. On the main result's open set, entry rises, high-class ownership rises when a prepared high-class rival surely beats the strong incumbent, and access to prices raises expected proceeds and net acquisition surplus against the strong incumbent. Under the paper's default tie rules, uninformative and informative markets coexist exactly on a region of the spread and preparation threshold, informative entry rising in the spread. At an even prior, where an indifferent informed trader may mix and the threshold lies above the prior, research and entry rise with the spread until research caps in the equilibrium in which a purchase after successful research recruits.

1 Introduction

At fixed information, a stronger incumbent bidder reduces a potential rival bidder's expected return from entry into a takeover contest. The rival must pay preparation costs before it can submit an executable offer, and stronger competition lowers what that preparation earns (**??**). I study a force in the opposite direction. Competition can make the target's stock more informative about the rival, and when the rival observes that price before it commits to acquisition preparation, the information effect can outweigh the loss of acquisition profits. A stronger incumbent can then induce rival entry and raise the probability that a rival from the high class acquires the target.

The setting is a listed target with a public sale opportunity. The incumbent is already prepared, and the rival has not committed to the diligence that an executable proposal requires; Section 2.1 describes the decision interval in which the rival decides while the target's stock trades. Incumbent strength is the public distribution of the incumbent's value; a stronger incumbent shifts it upward, and the rival decides before any offer is announced.

The rival and target shareholders hold two claims on the same acquisition surplus: the rival keeps what remains if it prepares, bids, and wins, and target shareholders hold the traded claim, what the cash second-price auction pays them. The rival's value comes from one of two quality classes, high and low, the high class dominates the low class, and the classes fix the expected value gap between them. By Proposition 1, a stronger incumbent never raises the probability that the rival wins, and when the incumbent wins it pays the rival's bid, so at least as much of the gap reaches target proceeds; a rival that wins pays at least as much in expectation, so no more of the gap remains as acquisition profit at every belief. Target shares therefore become at least as sensitive to information about the rival at the same time as the rival's own return to preparation falls weakly.

An outside informed trader with no net position in the target knows the rival's class and trades on that sensitivity. The trading stage is one order slot, with the alternative-arrival structure of **?**: with some probability the informed trader fills the slot, and otherwise a liquidity trader fills it with a sale, a purchase, or no order. No one observes who placed the order, and a liquidity order can match the informed trader's direction, so no order reveals the class. Under every strategy of the informed trader the posterior after every order stays inside fixed bounds, and the informed trader keeps a residual advantage after every order; a larger gap in target proceeds can make that advantage worth the cost of trade. Target shareholders gain when the price recruits entry, because a holder through the sale receives sale proceeds, not the interim price.

The main result, Proposition 2, compares three economies on a nonempty open set of parameters: one with a weak incumbent, one with a strong incumbent, and one with a very strong incumbent against which recruitment stops. In each economy the equilibrium outcome is unique. With the weak incumbent, the informed trader's residual advantage cannot cover the trading cost, the price

reveals nothing, and only a zero-cost rival prepares. With the strong incumbent, the informed trader buys after good news and sells after bad news, and a purchase brings in a high-cost rival, so entry rises even though acquisition profit falls at the prior and at every other fixed belief that gives the high class positive weight. When a prepared high-class rival beats the strong incumbent for sure, the probability that a rival from the high class acquires the target rises with entry; if, in addition, a low-class rival beats the weak incumbent with positive probability, the high class also wins a larger share of the takeovers that go to a rival. With the very strong incumbent, the price stays informative, but even the most favorable price fails to justify preparation at the high cost, so entry returns to its floor.

The reversal comes from the change in information that competition induces. Competitive market makers anticipate the preparation that each price induces, so the reversal holds under rational pricing; when the informed trader's orders stay fixed at their informative level while the incumbent strengthens, a stronger incumbent cannot raise entry (Section 4.2).

As an illustration, in the example of Section 4.2, entry rises from one quarter at the weak incumbent to 0.522757 at the strong incumbent, high-class ownership rises from one eighth to 0.324401, and the high class's share of the takeovers that go to a rival rises from 0.545455 to 0.830691. Over the same change, gross acquisition profit at the prior falls from 4.804167 to 4.291667, while the target-payoff spread rises from 1/60 to 2/3. In the example, high-cost preparation stops above the entry ceiling, the strength $r_C \approx 3.592659$. At the example's auction and costs and the prior $1/2$, the hypotheses of the main result hold exactly when $82/59 < u < 2686/1559$, where u is the informed presence mass relative to the liquidity mass in one direction; equivalently, the top posterior at the prior $1/2$ lies between the preparation thresholds of the strong and the very strong incumbent, $141/200$ and $4245/5804$ (Corollary 1).

When target proceeds and acquisition profit are both higher with a high-class rival, trading and preparation depend on the auction only through the target-payoff spread, which a stronger incumbent never narrows, and the preparation threshold (Section 3.4). Under the paper's default tie rules, an uninformative market and an informative market coexist exactly when the threshold lies above the prior and within reach of a purchase, with the spread large enough for a threshold-reaching purchase to cover the trading cost but small enough that no order pays against an uninformative price (Proposition 3; Section 4.3). Entry in the informative market moves only with the spread, rising while class H mixes and staying constant once class H buys for sure. Coexistence with a sure purchase forms one interval along any strengthening path, and the whole coexistence set is one interval when class H's gross profit exceeds the high cost against the weakest incumbent and the product of the spread and that excess does not fall. As an illustration, in the example at the prior $1/2$, coexistence runs from $r_A \approx 1.393958$ to $r_N \approx 1.747878$, with class H buying for sure from $r_F \approx 1.467127$ (Corollary 2).

When the informed trader can pay for research, choosing the probability of success up to a cap at a quadratic cost, research and entry rise together with the target-payoff spread until research reaches its cap in the equilibrium in which a purchase after successful research recruits a high-cost rival, under the paper's default tie rules at an even prior (Proposition 4; Section 5). Where the spread exceeds twice the trading cost and the preparation threshold lies above the prior, this equilibrium exists exactly when the posterior after a purchase, at the research the informed trader chooses, reaches the threshold. Selling on bad news leaves the rise in place: the low class abstains while a sale cannot pay its cost, then sells just often enough to break even, adding nothing to the value of research; a paying sure sale adds a second return. When class H outbids the incumbent, high-class ownership rises with research. As an illustration, in the example the recruiting equilibrium has research rising from 0.237901 at the strength 1.55 to 0.316683 at 1.65 and entry from 0.507630 to 0.519763, and zero research is also an equilibrium at those strengths (Corollary 3).

The rise in entry survives other distributions of preparation costs, with or without atoms, when every low cost lies below gross profit at the bottom posterior against the strong incumbent and every high cost lies between gross profit at the prior against the weak incumbent and gross profit at the top posterior against the strong incumbent (Section 6.1). For uniform incumbents, every prior and every pair of classes that satisfy the class ranking and differ above the reserve admit incumbent strengths and costs at which the reversal holds (Section 6.2); with more than two classes, it holds under a separation condition. At the prior $1/2$, it also survives private signals that the rival and the informed trader hold about different facts, under conditions on the costs (Section 6.3); in the signal example, with its own costs and strengths, the rival's signal is more accurate than the informed trader's, and the rise in that example needs both signals.

Under the hypotheses of Proposition 2, access to prices raises expected target proceeds and acquisition surplus net of preparation costs against the strong incumbent (Section 7.1). Sale terms shape both what a bidder pays when it wins and what the stock price can reveal before anyone prepares. With a single value in each class and the incumbent's highest value between them, an alternative bargaining institution shows how the payment rule matters: the effect of competition on the target-payoff spread depends on the weight the rule places on the runner-up's value (Section 7.2). A reserve that excludes the low class can make the price informative against both incumbents, and if informed trade is then full and only a purchase recruits at both strengths, entry is flat in incumbent strength (Section 7.3).

When the informed trader always buys after good news and always sells after bad news and the prior lies below the preparation threshold, a purchase raises the price by more than a sale lowers it, once each impact is weighted by how often its order occurs, exactly when the purchase recruits the rival; the test needs only prices and order frequencies (Section 8). The difference comes from the entry component of the price gap between a purchase and a sale; in the example at the strong

incumbent, that component is 90.2% of the price gap. Under the same trading pattern, a prepared rival wins more often after a purchase than after a sale whenever the incumbent's value sometimes falls where a high-class value beats it with a higher probability than a low-class value, and where entry stays at its floor after every order, orders predict the winner under the same condition, although they do not predict entry. Under the same trading pattern, order counts identify the informed trader's presence when the prior is known and differs from one half, and at the prior $1/2$ the sale impact identifies it, given the payoff scale and entry at its floor after a sale and after no order.

The paper belongs to the literature on the real effects of financial markets (**?**). The closest antecedent is **?**, where a firm's investment decision feeds back into the incentive to produce information about its stock; here the sale rule divides acquisition surplus between a traded claim and a rival that decides whether to prepare. **?** show that corrective real decisions can discourage trade on bad news under rational pricing; here the real decision is the rival's entry itself.

Firms learn from peers' stock prices (**?**), and target prices affect takeover activity (**?**); in this model, the rival learns about its own acquisition value from the target's price before it commits.[1]

The auction-entry literature makes the bidder pool endogenous (**????**). Under selective entry, a bidder's pre-entry information indexes the distribution of its value, and the payment of the entry cost reveals the value (**?**).[2] Here the pre-entry information is the target's stock price, a public price that moves with the orders of an informed trader who observes the rival's quality class. I keep the direct deterrence force of the auction-entry literature and add a market that operates before preparation, with the informed trader outside the auction and trading a payoff that differs from the rival's profit.[3]

The informed trader in this paper is an outsider and earns only the trading residual. In other work, an informed party can also hold a stake before a control decision: an acquirer can use liquidity trade to conceal share purchases before a bid (**?**), large holders trade on their information (**?**), and arbitrageurs' positions can affect tendering (**?**).[4]

Section 2 sets out the model, and Section 3 derives the returns to information and the two numbers through which trading sees the auction. Section 4 states the main results and introduces the example. Sections 5 to 8 cover research, robustness, welfare and sale terms, and empirical implications, and Section 9 concludes. Appendix A states Propositions 3 and 4, the three corollaries

[1] Bidders also learn from announcement returns when they decide whether to complete a deal (**?**). That decision comes after the entry margin of this paper.

[2] One limit of this framework is a bidder that knows only the common value distribution before entry (**?**). The other limit is a bidder that already knows its value (**?**). Selective entry also affects the choice of sale procedure (**?**), and auction formats affect bidders' incentives to acquire information (**?**).

[3] Recent work studies how bidders learn about their own values and about competitors (**?**), post-auction feedback in security-payment design (**?**), and bidder-pool choice with correlated values (**?**). In takeovers, **?** study negotiations with stock-market feedback, and **?** model payment choice and trade within an initiated deal; I hold cash consideration fixed and study entry.

[4] Large holders also monitor or intervene as prices feed back to firm decisions (**?**, Section II.E; **?**; **?**). **?** study manipulation through the real decision that responds to the price.

and the results of Sections 6 to 8 in full, and proves every result.

2 The model

2.1 The decision interval

The model applies to a listed target whose sale opportunity is publicly understood while its stock still trades and while a further bidder can still come in. A disclosed approach, an announced strategic review, or an open contest can create such an interval if the opportunity becomes public before the set of bidders is fixed; only then can a further bidder watch the price during the period that matters. Section 8 states how to date this interval in a given transaction.

Within the interval, one bidder is already prepared; I call it the incumbent bidder, or the incumbent. It is the party that approached the target or a bidder that has completed diligence, and its strength is what the public can infer about its acquisition value from the disclosed record. A second bidder, the potential rival bidder, or the rival, has not decided whether to prepare. Preparation is required for an executable proposal: before a bidder can submit a binding offer, it must verify the target's contracts, liabilities, and technology, arrange financing and approvals, and commit evaluation capacity that has other uses. Those costs are sunk once incurred, and they differ across bidders, since prior familiarity, capacity already in place, or a small deal relative to the bidder's size make preparation cheap for some bidders and expensive for others. A rival that does not prepare stays out of the sale, so its cost is an entry cost: payment reveals its value and permits a bid.

The informed trader and the rival know different things. The rival knows its own integration plans, while an informed trader with sector expertise can know what the target's customers do, how its technology compares with alternatives, or how its product demand will change. These are different facts about the same acquisition match, so a more accurate signal about one can leave the other unknown. The model gives the informed trader exact knowledge of the rival's quality class and no further news about its value, and suppresses the rival's initial information; Section 6.3 removes both extremes, and Section 5 makes the informed trader's information the result of costly research. The identity of the rival is common knowledge, and the market is uncertain only about how well the rival fits the target.

Public incumbent strength is a distribution, not a bid. The record can reveal that a strategic acquirer with clear synergies is at the table, or that the prepared party is a financial buyer with a leveraged capital structure, and in each case it can stay silent on what either bidder would pay; strategic and financial bidders differ systematically in their valuations (**?**). The model varies that distribution and keeps the incumbent's realized value private until it bids. Disclosure records also document a costly preparation stage: Imprivata's definitive proxy separates an unsolicited approach,

outreach to potential buyers, and indications of interest conditional on further diligence (**?**), and takeover competition develops before public bids (**??**). Section 8 states how to test whether prices draw bidders into that preparation stage.

2.2 Values, preparation, and the sale

The target's known standalone value is normalized to zero, and acquisition values and preparation costs are measured per target share. A common standalone component added to every ownership outcome shifts prices and payoffs by a constant and leaves incentives unchanged.

All strategic agents are risk neutral. The target share is the only traded claim with information about the acquisition match, and neither bidder trades it. The sale mechanism binds all target shares, so shareholder tendering and holdout stay outside the model. The incumbent's value distribution is public when the sale opportunity becomes visible.

The target faces two potential acquirers. The incumbent has already prepared, incurs no further entry cost, and knows its value before it bids. Its acquisition value R has a continuous distribution F on $[0, \bar{r}]$, conditional on public information when the sale process begins, and F is extended by unity above $\bar{r}$. A first-order stochastic shift of F strengthens the incumbent. A path is a family $\{F_r\}$ of continuous incumbent distributions indexed by a strength r in an interval, in which $F_{r'}$ is a first-order stochastic strengthening of F_r whenever $r' > r$, with everything else fixed.

The rival belongs to one of two quality classes, $j \in \{H, L\}$, with prior $\pi = \Pr(j = H) \in (0, 1)$. Given its class, its acquisition value θ has a distribution G_j on $[0, \infty)$ with a finite mean; atoms are allowed. The probability that a class-j value falls below the reserve p is $G_j(p^-)$, and $D(y) = G_L(y) - G_H(y)$ is the difference between the two class distributions at y.

The high class dominates the low class in the sense of first-order stochastic dominance. Both class supports lie above the reserve, so that every value meets it:

$$G_H(y) \leq G_L(y) \quad \text{for every } y, \qquad G_H(p^-) = G_L(p^-) = 0. \tag{1}$$

I call the two restrictions in (1) the class ranking.

Under the class ranking (1), competition never narrows the class difference in target proceeds, whatever the incumbent distribution (Proposition 1). Appendix A.1 characterizes the weakest restriction of this kind: when neither class has a value atom at the reserve, that difference rises under every strengthening exactly when the high class dominates above the reserve and the two classes fall below it with equal probability, which I call equal admissibility. Section 7.3 studies a reserve that excludes the low class, where equal admissibility fails. The structure is the selective-entry structure of the auction literature: the class indexes the distribution of the rival's value, preparation reveals the value, and the stock price is the pre-entry signal.

The general statements need no normalization of money: a common positive rescale of every value, the reserve, and every preparation, trading and research cost rescales prices and payoffs by the same factor, while orders, beliefs, entry, and ownership do not change.

The rival initially knows neither its class nor its value. After it observes the stock price, it privately learns its preparation cost $C \in \{0, c_H\}$, with $\Pr(C = 0) = \rho$, $0 < \rho < 1$, and $c_H > 0$. The low cost is zero: a zero-cost rival is a ready bidder whose diligence costs nothing at the margin. A payment of C reveals θ and permits a bid, and a rival that does not pay stays outside the sale.

The pair (j, θ), the incumbent's value, the preparation cost, and the draws of the order slot in Section 2.3 are mutually independent. The cost therefore carries no information about the class or the value, and conditional on the class, the rival's value is independent of the order.

Before trade, the seller publicly commits to a cash second-price auction with reserve $p > 0$. The highest admissible bidder acquires the target and pays the larger of the reserve and the highest competing admissible bid. A bid equal to the reserve is admissible, and there is no sale without an admissible bid. Each prepared bidder whose value meets the reserve bids its value; this bid is weakly dominant (Appendix A.1).

A prepared rival whose value is below the reserve does not bid. Without a rival, the incumbent buys at the reserve when its value meets it, so expected no-entry proceeds are $t_0 = p[1 - F(p)]$: the board collects its announced walk-away price from a lone prepared bidder whose value meets it. Because F is continuous, acquisition-value ties have probability zero.

2.3 Trading: the order slot

The informed trader observes the rival's class j. It is an outside party, such as a sector-focused long-short fund. It holds no net position in the target before trade, has no control rights, and cannot acquire the target. Its information concerns the rival's fit with the target, and it forms that view from its own research on the target's customers, technology, and demand.

Trade takes place in one order slot, a single public order opportunity that comes before the rival's decision. With probability $\lambda \in (0, 1)$ the informed trader fills the slot: it sees the class j and chooses an order $q \in \{-1, 0, 1\}$, and a nonzero order costs $k > 0$. With probability $1 - \lambda$ a liquidity trader fills the slot instead and, for exogenous reasons apart from the sale, sells, abstains, or buys one unit with probabilities β, $1 - 2\beta$, and β, where $0 < \beta \leq 1/3$.[5] A liquidity sale comes from a current shareholder who sells for cash, a liquidity purchase from an outside investor, and an abstention means that the opportunity passes with no order. No one observes who filled the slot, and the identity draw is independent of the class, the costs, and the incumbent's value.

[5]The posterior bound common to all orders, and through it the exclusion bound of Section 3.2, use $\beta \leq 1/3$. The research parameter region and the slot-feasibility statements in identification range over the slots of the model. The other results hold for every $0 < \beta < 1/2$ (Appendix A.2).

Competitive market makers see the order $X \in \{-1, 0, 1\}$, set the price $P = \mathbb{E}[V_T \mid X]$, where V_T is the terminal payoff of a target share, and absorb the order at that price; each share held at the sale, market-maker inventory included, receives V_T. If the informed trader fills the slot with order q, its order is the whole order that market makers see, so it trades at the price $P(q)$ and earns

$$q\,[V_T - P(q)] - k|q|. \tag{2}$$

The unit is a normalized small trading unit. The cost k is a separate friction, such as an execution cost or a cost to carry the position, and it is distinct from adverse-selection price impact, which the competitive price already generates. The informed trader's order after bad news is a short sale, and k includes the cost to borrow shares.

A liquidity order in a given direction arrives with probability $b = (1-\lambda)\beta$, and the ratio $u = \lambda/b$ is the informed presence mass relative to the liquidity mass in one direction; the number $1 + u$ is the largest likelihood ratio that one order can carry. Section 3.2 shows that, under every strategy of the informed trader, the posterior after every order lies between a bottom posterior $m(\pi)$ and a top posterior $M(\pi)$:

$$M(\pi) = \frac{\pi(1+u)}{\pi(1+u) + 1 - \pi}, \qquad m(\pi) = \frac{\pi}{\pi + (1-\pi)(1+u)}. \tag{3}$$

The bounds follow from one property of the slot: each class reaches each order through a liquidity trader, so the informed trader keeps a residual advantage after every order.

The slot uses the alternative-arrival structure of **?**, in which an informed order and a liquidity order of the same size are alternative arrivals in one opportunity; in **?**, a liquidity shock forces the informed holder's own sale, so a sale occurs on path for both motives. The unit order menu, the presence probability, and the symmetric unit cost follow **?**.[6] Risk-neutral market makers with zero expected profit price each order, as in **?** and **?**. Here the terminal payoff depends on the price through preparation.

Market makers observe the order X and nothing else. The rival observes the price P and its own cost C, but not X, its class, its value, or the incumbent's value; the informed trader observes the class j and no further news about the value θ, and the incumbent knows its own value. The sequence is as follows. The seller announces the sale rule. The informed trader learns the class, the slot draw selects the informed trader with probability λ and a liquidity trader otherwise, and the

[6] The unit menu $\{-1, 0, 1\}$ keeps every order on path, so no off-path belief enters the analysis. A menu of continuous order sizes needs a convention for the sizes that no class uses in equilibrium, as in **?**. Another route keeps every size on path through additive liquidity flow: with Laplace noise of scale ξ and full unit orders, the largest likelihood ratio of a favorable flow is $\mathrm{e}^{2/\xi}$, where e is Euler's number, printed upright to separate it from entry $e(x)$, so the top posterior equals the slot's $M(\pi)$ when $1 + u = \mathrm{e}^{2/\xi}$. The two set-ups then share the posterior bounds, and they differ in the probability with which each class reaches the top posterior.

selected trader places its order. Market makers set the price from the order, and the rival observes the price and its cost and decides whether to prepare. A rival that prepares learns its value. The incumbent and a prepared rival then submit their bids, and ownership and financial payoffs are realized. Section 5 adds a research stage before the informed trader learns the class.

2.4 Equilibrium

An equilibrium specifies, for each class, a distribution over the three orders; the market makers' posterior $\mu(x) = \Pr(j = H \mid X = x)$ and price $P(x)$ after each order x; the rival's belief at each price; its preparation choice at each price and cost; and truthful bids. The informed trader can mix, and wrong-signed orders are allowed. Every order has positive probability in both classes, because a liquidity trader can place it, so Bayes' rule gives a posterior after every order. Prices are competitive. A high-cost rival prepares when gross profit at its belief exceeds c_H and stays out when it falls short, and a zero-cost rival, whose gross profit is an expected auction rent and never negative, prepares after every price. The informed trader's order maximizes

$$U_j(q) = q\left[t_0 + e(q)(t_j - t_0) - P(q)\right] - k|q|$$

at the candidate's prices and entry, where $e(q)$ is entry after the order q, the probability that the rival prepares after the price $P(q)$; t_j is expected target proceeds after entry in class j, and t_0 is expected target proceeds without entry. Definition A.1 in Appendix A states the six conditions in full.

For a known value θ of the rival, expected target proceeds after the rival prepares are $t(\theta)$, and the rival's expected gross acquisition profit is $g(\theta)$; as before, t_0 is expected target proceeds without entry. The class objects are expectations over the class distribution, $t_j = \mathbb{E}_j[t(\theta)]$ and $g_j = \mathbb{E}_j[g(\theta)]$ for $j \in \{H, L\}$, and the target-payoff spread is $\Delta_T = t_H - t_L$. Bids are truthful, so these objects depend on the incumbent's distribution and the reserve but not on any belief. At a belief $\mu = \Pr(j = H)$, the rival's expected gross profit is $B(\mu) = g_L + \mu\,(g_H - g_L)$, because its value depends on the price only through its class. When the incumbent's distribution matters, I write $g_j(F)$, $\Delta_T(F)$, and B_F; along a path, I write $g_j(r)$, $\Delta_T(r)$, and B_r for the objects at F_r.

The informed trader's objective takes the expected terminal payoff of a target share given class j and order q as $t_0 + e(q)(t_j - t_0)$, because entry depends only on the price and the cost, and the cost is independent of the class and the order. The same independence, with Bayes' rule, turns the competitive price into

$$P(x) = t_0 + e(x)\left[t_L - t_0 + \Delta_T\,\mu(x)\right]. \tag{4}$$

The ex ante payoff of the informed trader multiplies U_j by λ, and this factor leaves the best order unchanged.

When ties arise, an indifferent informed trader may mix over its best orders, and an indifferent high-cost rival prepares. I call these the default tie rules; the other two are that an indifferent informed trader abstains and that an indifferent high-cost rival may prepare with any probability. Where another tie rule changes a conclusion, the appendix states the change beside the result, and Table A.3 in Appendix A.4 collects the changes for coexistence in the example at the prior $1/2$.

A unilateral deviation by the informed trader changes only its own order and faces the candidate equilibrium's price function and entry rule, so the deviation payoff is $U_j(q)$ read at the candidate's $P(q)$ and $e(q)$. A comparison across economies, such as across incumbent strengths, solves for the equilibrium again in each economy.

Entry E is the probability that the rival prepares, and high-class ownership O_H is the probability that a rival of class H acquires the target. The probability that a prepared rival of class j wins against the incumbent distribution F is $W_j(F) = \mathbb{E}_j\big[F(\theta)\mathbf{1}\{\theta \geq p\}\big]$, and class H outbids the incumbent F when $W_H(F) = 1$, so that a prepared class-H rival beats it for sure. Entry in class j, e_j, is the probability that the rival prepares given class j, and class H's takeover share is the probability that the rival is of class H, given that the rival acquires the target.

3 Competition and information

3.1 Acquisition profits and target value

The auction determines both the return to preparation and the claim that informed traders trade. Consider a prepared rival with value $\theta \geq p$. If it wins, it pays $\max\{p, R\}$; if it loses, the incumbent pays θ. The target therefore receives $\max\{p, \min(R, \theta)\}$, and the rival keeps $(\theta - \max\{p, R\})_+$. A prepared rival with $\theta < p$ cannot bid, and the target then receives the reserve if the incumbent's value meets it, exactly as without entry. For every value at or above the reserve, the two expected amounts add up to the value:

$$t(\theta) + g(\theta) = \theta, \qquad \theta \geq p.$$

The second-price rule passes the whole value of an admissible rival to the target and the rival together. An admissible value also raises proceeds to at least the reserve, so $t_j \geq t_0$ for both classes.

A stronger incumbent never raises the probability that a prepared rival wins, and when the incumbent wins it pays the rival's bid. Proposition 1 shows that it weakly lowers acquisition profit at every fixed belief and weakly raises the sensitivity of target proceeds to the quality of the rival, for every continuous incumbent distribution; the second effect uses the class ranking (1).

Proposition 1 (Competition and the two returns to information)**.** *Let the incumbent's value have a continuous distribution F on $[0, \bar{r}]$, extended by unity above $\bar{r}$. For any classes G_H, G_L with finite*

means,

$$\begin{aligned} \Delta_T(F) &= pF(p)D(p^-) + \int_p^\infty [1 - F(y)]\, D(y)\, dy, \\ g_j(F) &= \int_p^\infty [1 - G_j(y)]\, F(y)\, dy, \qquad j \in \{H, L\}. \end{aligned} \tag{5}$$

Let $\tilde{F} \leq F$ be a continuous first-order stochastic strengthening of F.

(a) For every pair of classes, the strengthening weakly decreases g_H and g_L, and hence gross acquisition profit at every fixed belief.

(b) Under the class ranking (1), the strengthening weakly increases the target-payoff spread.

The decrease in g_j is strict when $\int_p^\infty [1 - G_j(y)][F(y) - \tilde{F}(y)]\, dy > 0$. The increase in (b) is strict when $\int_p^\infty [F(y) - \tilde{F}(y)]\, D(y)\, dy > 0$.

At one incumbent value R that meets the reserve, the target receives $\min\{R, \max\{\theta, p\}\}$, and the expected class difference in proceeds is $\int_p^R D(y)\, dy$. Each slice $D(y)\, dy$ of the class value gap at a level $y > p$ reaches the target when the incumbent outbids the rival at that level and pays its bid, and it stays with the rival otherwise.

A stronger incumbent moves slices to the target, and under the class ranking it never narrows the spread and never raises acquisition profit. The reserve adds one term: below the reserve, only an admissible rival buys, at p, and the class difference is $pD(p^-)$. A stronger incumbent fails the reserve less often, so that term shrinks, and it can outweigh the moved slices when the low class is more often inadmissible. Under the class ranking (1), every value meets the reserve, and the term is zero. Appendix A.1 gives the proof and adds a part (c): when neither class has a value atom at the reserve, the spread weakly increases under every strengthening of every continuous incumbent distribution if and only if the high class dominates above the reserve and the classes have equal admissibility (Section 2.2).

The sum of the two lines of (5) under the class ranking (1) is

$$\Delta_T(F) + g_H(F) - g_L(F) = \mathbb{E}_H[\max\{\theta, p\}] - \mathbb{E}_L[\max\{\theta, p\}].$$

The identity holds for every incumbent distribution, and its right side does not depend on the incumbent. A stronger incumbent leaves the value gap between the classes unchanged; it moves the gap from the rival's profit to target proceeds.

A stronger incumbent therefore never narrows the spread and never raises gross acquisition profit at any belief. Whether entry rises depends on how this change in payoffs affects the information in the price.

3.2 What the rival learns from the price

The rival sees one price. A liquidity order in the same direction is always possible, and this bounds what any order can reveal. Each class reaches a given nonzero order through a liquidity trader with probability b, and the informed trader adds at most λ, so the likelihood ratio of the order lies between $1/(1+u)$ and $1+u$. By Bayes' rule, the posterior after every order lies in the interval $[m(\pi), M(\pi)]$ of (3), under every strategy of the informed trader, mixed or pure; when the high class buys and the low class sells, the posteriors after a sale, no order, and a purchase are $m(\pi)$, π, and $M(\pi)$. The rival's belief at a price averages the posteriors of the orders with that price, so it obeys the same bounds. No price can therefore support a preparation decision that needs a belief above the top posterior. This gives the exclusion bound: when $g_H \geq g_L$, gross profit does not fall with the belief, so once gross profit at the top posterior falls below c_H, no order recruits a high-cost rival, in or out of equilibrium.

When $g_H > g_L$, gross profit covers the high cost exactly when the belief reaches the preparation threshold $\tau = (c_H - g_L)/(g_H - g_L)$: $B(\mu) \geq c_H$ if and only if $\mu \geq \tau$. Along a path I write $\tau(r)$. A zero-cost rival prepares after every price, so entry after every order is at least ρ, and competitive pricing gives (4). When the classes differ in target proceeds, the price rises strictly with the posterior. Suppose two orders have one price. Entry after them is the same, because entry depends on the order only through the price, and at least ρ; the price formula then gives the two orders the same posterior. The rival therefore reads the market makers' posterior from the price, and its gross profit at the price $P(x)$ is $B(\mu(x))$, so high-cost preparation follows exactly the orders whose posterior reaches τ. Appendix A.2 gives the formal statements and their proofs.

The zero-cost rival gives each step its floor: entry stays at least ρ, so the price reveals the posterior, and the gross return of a correctly signed order stays at least a fixed share of the spread, the trading bound of Section 3.3. Its preparation does not respond to the price, so information can only add entry.

3.3 Informed trading

Rational pricing removes the anticipated increase in proceeds from the informed trader's advantage. What remains is the gross return of a purchase in class H and of a sale in class L,

$$K_H = e(1)\,\Delta_T\,[1-\mu(1)], \qquad K_L = e(-1)\,\Delta_T\,\mu(-1), \qquad \rho\,\kappa(\pi)\,\Delta_T \leq K_H,\, K_L \leq \Delta_T,$$

with $\kappa(\pi) = \min\{m(\pi), 1-M(\pi)\}$. A wrong-signed order earns at most $-k$, so it never pays. Class H buys when K_H exceeds k and abstains when it falls short, and class L compares K_L with k in the same way.

Market makers price the entry response, and the posterior bounds of Section 3.2 leave some uncertainty about the class after every order. The informed trader's gross return is the entry probability times the spread times that residual uncertainty. The lower bound uses the entry floor ρ and the posterior bounds, and the spread bounds the return from above. Both bounds hold under every candidate profile, mixed or pure, and their comparison with k decides when information survives in equilibrium.

Since $1 - M(\pi) = m(1 - \pi)$, the bound takes the form $\kappa(\pi) = m(\min\{\pi, 1 - \pi\})$. It is largest at the prior $1/2$, where it equals $m(1/2) = 1/(2 + u)$, and smaller at every other prior. The class that the prior favors binds, because the price already leans toward it.

The no-entry proceeds t_0 cancel from each return, because they move the price and the terminal payoff together; the terms that matter are the entry gains $t_H - t_0$ and $t_L - t_0$, which are nonnegative.

Each return depends only on the posterior and entry after its own order, so entry after the zero order enters no payoff, and the informed trader's problem splits by class. Appendix A.2 gives the formal statement and proof of the trading bound, and the coexistence results of Section 4.3 rest on this split.

3.4 What the market sees of the auction

When the spread is positive and class H's gross profit exceeds class L's, the informed trader's orders, the market makers' posteriors and the rival's preparation depend on the auction only through the target-payoff spread Δ_T and the preparation threshold τ. The spread sets what an informed order earns at each posterior and level of entry, and the threshold sets which posteriors recruit a high-cost rival. Two auctions with the same spread and the same threshold, and with every other primitive and the tie rules the same, therefore have the same equilibrium orders, posteriors and entry, and, when the informed trader chooses research, the same research. Prices add the levels t_0 and t_L, and ownership adds which bidder wins, so results on ownership carry class H's chance of beating the incumbent.

A stronger incumbent never narrows the spread (Proposition 1). It never lowers the threshold while a rival known to be in class L earns at most its high preparation cost, $g_L \le c_H$: it never raises class L's profit and never widens the profit gap between the classes, so a rival needs at least as favorable a belief before preparation at the high cost pays. Along a path, once $g_L \le c_H$ holds, it holds against every stronger incumbent. Coexistence in Section 4.3 and research in Section 5 are statements about these two numbers, and Appendix A.2 gives the formal statements and their proofs.

4 Equilibrium results

4.1 Competition creates competition

The main result compares three economies that differ only in incumbent strength. Against the weak incumbent the price carries no information and only a zero-cost rival prepares; against the strong incumbent the price is informative and a purchase recruits a high-cost rival; against the very strong incumbent the price stays informative, while entry returns to its floor. I write F_0, F_1 and F_2 for the three incumbent distributions.

Proposition 2 (Competition creates competition)**.** *Let the classes satisfy the class ranking* (1) *and have finite means, let F_1 be a continuous first-order stochastic strengthening of the continuous incumbent distribution F_0, and let F_2 strengthen F_1 in the same sense. Fix $0 < \pi, \rho, \lambda < 1$, $0 < \beta < 1/2$, $c_H > 0$ and $k > 0$. Write $\Delta_i = \Delta_T(F_i)$, $B_i = B_{F_i}$, and $t_{0,i}$ and $t_{L,i}$ for the proceeds t_0 and t_L against F_i, with $b = (1-\lambda)\beta$ and $\kappa(\pi) = \min\{m(\pi), 1 - M(\pi)\}$. Suppose that*

$$\begin{aligned} &\textit{(H1), the high-cost window:} && B_0(\pi) < c_H < B_1(M(\pi)), \\ &\textit{(H2), the trading-cost window:} && \Delta_0 < k < \rho\kappa(\pi)\Delta_1, \\ &\textit{(H3), the exclusion condition:} && B_2(M(\pi)) < c_H. \end{aligned}$$

Each economy has a unique equilibrium outcome under all four tie rules.

(i) The weak incumbent F_0. *Both classes abstain, every posterior equals π, entry is ρ, and all orders share the price $t_{0,0} + \rho[t_{L,0} - t_{0,0} + \Delta_0\pi]$. High-class ownership is $\pi\rho W_H(F_0)$.*

(ii) The strong incumbent F_1. *Class H buys and class L sells with probability one. The posteriors after $(-1, 0, 1)$ are $(m(\pi), \pi, M(\pi))$, and entry is $(\rho, \rho, 1)$. Prices rise strictly in this order. Class entry and total entry are*

$$e_H = \rho + (1-\rho)(b + \lambda), \qquad e_L = \rho + (1-\rho)b, \qquad \mathsf{E}_1 = \rho + (1-\rho)(b + \pi\lambda) > \rho.$$

High-class ownership rises exactly when $e_H W_H(F_1) > \rho W_H(F_0)$. Class H's takeover share rises exactly when

$$e_H W_H(F_1) W_L(F_0) > e_L W_H(F_0) W_L(F_1). \tag{6}$$

If class H outbids the stronger incumbent, $W_H(F_1) = 1$, it also outbids F_0; high-class ownership rises from $\pi\rho$ to πe_H, and its takeover share is nondecreasing, strictly exactly when $W_L(F_0) > 0$. If $W_L(F_0) = 0$, that share is one against both incumbents.

(iii) The very strong incumbent F_2. *The orders and posteriors are those of part (ii), entry is ρ after every order, and the three prices differ. High-class ownership is $\pi\rho W_H(F_2)$.*

Against each F_i, with $\mu_i(x)$ and $e_i(x)$ the posterior and entry after the order x, prices are $P_i(x) = t_{0,i} + e_i(x)[t_{L,i} - t_{0,i} + \Delta_i \mu_i(x)]$, the price belief is $\mu_i(x)$, high-cost preparation is $[e_i(x) - \rho]/(1 - \rho)$, and bids are truthful.

The high-cost window (H1) sets entry through acquisition profit: gross profit at the prior falls short of the high cost against F_0, so an uninformative price recruits no high-cost rival, while gross profit at the top posterior exceeds the high cost against F_1, so a purchase can recruit one. The trading-cost window (H2) fixes trading before any order profile is known: by the trading bound of Section 3.3, every correct order earns at most $\Delta_T(F_0) - k < 0$ against the weak incumbent and at least $\rho\kappa(\pi)\Delta_T(F_1) - k > 0$ against the strong incumbent, so a correct order cannot cover the trading cost in the first economy and covers it in the second. The exclusion condition (H3) stops recruitment against F_2, where gross profit at the top posterior falls short of the high cost (the exclusion bound of Section 3.2).

The trading-cost window is nonempty exactly when $\Delta_T(F_0)/\Delta_T(F_1) < \rho\kappa(\pi)$, that is, when competition widens the target-payoff spread enough between the two incumbent distributions. The condition involves the two distributions, the classes, ρ, the order slot and π, and no unit of money, and a prior away from one half narrows the window, since $\kappa(\pi)$ falls when the price already leans toward the favored class. Section 4.2 shows that (H1)–(H3) hold together on an open set of parameters.

The additional entry favors the high class, which places a purchase with probability $b + \lambda$, while the low class places one with probability b. High-class ownership multiplies entry in class H by class H's chance of beating the incumbent, and a stronger incumbent can lower that chance when class H can lose. The ownership condition in part (ii) holds whenever the stronger incumbent leaves the high class a good chance to win, $(1 + \rho)W_H(F_1) \geq 2\rho W_H(F_0) > 0$, and the high class places a purchase more often than not, $b + \lambda > 1/2$. When class H outbids F_1, its chance to win is one against both incumbents, so ownership rises with entry in class H; the stronger incumbent then takes wins only from the low class, and when a low-class rival can beat the weak incumbent, the takeovers that go to the rival shift toward class H.

Values and costs alone give outbidding when class H has a decisive advantage on a family of incumbent distributions: against every distribution in the family at which class H's gross profit exceeds c_H, class H outbids the incumbent. When $g_H > g_L$, a price recruits a high-cost rival only where class H's gross profit exceeds c_H, because gross profit at every posterior below one falls short of g_H. Under a decisive advantage, (H1) therefore gives $W_H(F_1) = 1$, and the takeover-share statement of part (ii) applies.

Part (iii) completes the pattern, in which entry first rises and later falls as the incumbent strengthens. Against the very strong incumbent even the top posterior fails to justify high-cost preparation, while trading stays informative, and Section 4.3 describes where uninformative and informative markets coexist between the three economies. Appendix A.3 gives the proof and the

three equilibrium tuples. Every comparison behind the result is strict, so the unique outcome in each economy is the same however ties are broken.

4.2 The example and the fixed-information control

The numbers and figures of the paper come from an example. In it, the two classes are point masses: G_H puts all its mass on $h = 10$ and G_L on $\ell = 1$, so the class and the value coincide. The incumbent is uniform on $[0, r]$, so an increase in r strengthens it and the strengths form a path. The prior is $\pi = 1/2$, the reserve $p = 1/2$, the zero-cost probability $\rho = 1/4$, the high cost $c_H = 6$ and the trading cost $k = 1/50$, and money is measured in units of the low value. The weak, the strong and the very strong incumbent have the strengths $r_0 = 6/5$, $r_1 = 3$ and $r_2 = 18/5$. Values, the reserve and every strength of the path satisfy

$$0 < p < \ell < r < h. \tag{7}$$

In the example's slot the largest likelihood ratio of an order is $1 + u = \mathrm{e}$, and the liquidity probability is $\beta = 0.294655$, a value inside the model's domain $\beta \leq 1/3$. At the prior $1/2$, whether (H1)–(H3) hold depends on the slot only through u (Corollary 1), and β then fixes the presence probability $\lambda = u\beta/(1 + u\beta) = 0.336122$, with $b = 0.195615$, and the posterior bounds at the prior are $M(1/2) = \mathrm{e}/(1 + \mathrm{e}) = 0.731059$ and $m(1/2) = 0.268941$; Appendix A.9 lists these values and the arithmetic behind each. With the example's classes and reserve, (H1)–(H3) are strict inequalities between continuous functions of $(r_0, r_1, r_2, c_H, k, \rho, \pi, \lambda, \beta)$, so they hold on an open set of these parameters that contains the example.

Corollary 1 (Recruitment and ownership in the example)**.** *In the example at the prior* $1/2$*, (H1)–(H3) hold if and only if* $82/59 < u < 2686/1559$*, that is, if and only if the top posterior lies between the preparation thresholds of the strong and the very strong incumbent,* $\tau(r_1) = 141/200 < M(1/2) < 4245/5804 = \tau(r_2)$*. When* $1 + u = \mathrm{e}$*, as in the example's slot, they hold if and only if the prior lies between* ≈ 0.467850 *and* ≈ 0.500424*. On this set class H outbids every incumbent on the path, entry rises from* ρ *at* r_0 *to* $\rho + (1 - \rho)(b + \pi\lambda)$ *at* r_1 *and returns to* ρ *at* r_2*, and high-class ownership rises from* $\pi\rho$ *to* $\pi[\rho + (1 - \rho)(b + \lambda)]$.

Table 1 reports the acquisition payoffs, and the two forces move in opposite directions from r_0 to r_1: gross acquisition profit at the prior falls from 4.804167 to 4.291667, while the target-payoff spread rises from 0.016667 to 0.666667. In the example's slot, entry rises from 0.250000 at r_0 to 0.522757 at r_1, and high-class ownership from 0.125000 to 0.324401. Class H outbids every incumbent on the path, because h exceeds r_2, and the strong incumbent takes wins only from the low class, so class H's takeover share rises from 0.545455 to 0.830691. At r_2 entry returns to

Table 1. Acquisition payoffs in the example.

	$r_0 = 6/5$	$r_1 = 3$	$r_2 = 18/5$
Proceeds without entry, t_0	0.291667	0.416667	0.430556
Proceeds with a high-class rival, t_H	0.704167	1.541667	1.834722
Proceeds with a low-class rival, t_L	0.687500	0.875000	0.895833
Target-payoff spread, Δ_T	0.016667	0.666667	0.938889
Gross profit of class H, g_H	9.295833	8.458333	8.165278
Gross profit of class L, g_L	0.312500	0.125000	0.104167
Gross profit at the prior, $B_r(1/2)$	4.804167	4.291667	4.134722
Preparation threshold, $\tau(r)$	0.633117	0.705000	0.731392

Notes. Point-mass classes h and ℓ, reserve p, incumbent uniform on $[0, r]$. The entries follow from equation (8). They do not depend on the slot.

0.250000 and the three prices still differ; Table 3 lists the three equilibrium tuples. With point-mass classes and a uniform incumbent of any strength, class H has a decisive advantage exactly when $c_H \geq (h^2 - p^2)/(2h)$, and the example's high cost exceeds $(h^2 - p^2)/(2h) = 399/80$: against every uniform incumbent at which a price recruits a high-cost rival, class H outbids the incumbent.

In the example, integration over R gives

$$\begin{aligned} t_0 &= p\Big(1 - \frac{p}{r}\Big), & t_H &= \frac{r}{2} + \frac{p^2}{2r}, \\ t_L &= \ell - \frac{\ell^2 - p^2}{2r}, & g_H &= h - \frac{r}{2} - \frac{p^2}{2r}, \\ g_L &= \frac{\ell^2 - p^2}{2r}, & \Delta_T(r) &= \frac{(r-\ell)^2}{2r}, \end{aligned} \tag{8}$$

with the derivatives

$$\Delta_T'(r) = \frac{1}{2} - \frac{\ell^2}{2r^2} > 0, \qquad g_H'(r) = -\frac{1}{2} + \frac{p^2}{2r^2} < 0, \qquad g_L'(r) = -\frac{\ell^2 - p^2}{2r^2} < 0, \tag{9}$$

and the preparation threshold

$$\tau(r) = \frac{2rc_H - (\ell^2 - p^2)}{2rh - r^2 - \ell^2}.$$

The right side of the class-gap identity of Section 3.1 is $h - \ell$ at every r. The spread approaches zero as r approaches ℓ, since the incumbent then almost never outbids a class-L rival, and Table 1 reports both numbers at the three strengths.

Appendix A.3 states Corollary 1 in full. Below the band a purchase at r_1 does not recruit a high-cost rival, and above it a purchase recruits at r_2, so the exclusion condition fails; far above it the trading-cost window fails at r_1. Table 2 gives the recruitment region in decimals, with the

Table 2. The recruitment region in the example.

Parameter	Region	Example
Mass ratio u	$(1.389831, 1.722899)$	1.718282
Top posterior $M(1/2)$	$(0.705000, 0.731392)$	0.731059
Presence λ at $\beta = 0.294655$	$(0.290539, 0.336721)$	0.336122
Presence λ at the largest β	$(0.316602, 0.364797)$	
Prior π, all three parts	$(0.467850, 0.500424)$	1/2
Prior π, parts (i) and (ii)	$(0.467850, 0.633117)$	1/2
Liquidity probability β	$(0, 1/3]$	0.294655

Notes. The example's auction and costs, strengths r_0, r_1, r_2. Rows 1 to 4 fix the prior at $1/2$; rows 5 and 6 fix $1 + u = \mathrm{e}$. Rows 1 to 5 give the ranges on which (H1)–(H3) hold, and row 6 gives the range for (H1)–(H2). Row 7 gives the model's liquidity domain at the example's likelihood-ratio bound. Decimal endpoints are rounded to six places; Corollary 1 in Appendix A.3 gives the exact endpoints, and Table A.6 in Appendix A.9 gives the example's distance to each boundary.

example's slot; Table A.6 in Appendix A.9 gives its distance to each boundary.

Both ends of the band are preparation thresholds: the lower end $\tau(r_1)$ is the belief that a purchase needs to recruit a high-cost rival against the strong incumbent, and the upper end $\tau(r_2)$ is the belief at which a purchase would still recruit one against the very strong incumbent. The threshold rises slowly with strength, and this sets the width of the band: class H outbids every incumbent on the path, and its expected payment $r/2 + p^2/(2r)$ rises by less than one half per unit of strength, a change that is small next to the gap $g_H - g_L$ in Table 1.

Gross profit at every belief falls strictly with strength above ℓ, so the exclusion condition (H3) holds at every strength in (r_C, h), where the entry ceiling $r_C \approx 3.592659$ is the strength at which $\tau(r) = M(1/2)$; parts (i) and (ii) of Proposition 2 hold without the exclusion condition. Table A.6 in Appendix A.9 records every hypothesis and its positive margin in the example.

Figure 1 shows the preparation threshold along the path: the prior lies below it at every strength, so an uninformative price never recruits a high-cost rival, and the threshold rises with strength until it meets the top posterior at the entry ceiling r_C, just below r_2. Panel (b) shows the band of Table 2 for the top posterior (Corollary 1).

Holding the information in the price fixed while the incumbent strengthens removes the reversal. The experiment fixes the distribution of the class and of a signal that the rival observes, the class value distributions and the cost distribution. Given the class, the signal tells the rival nothing more about either value. Its own value depends on the signal only through the class; the incumbent's value is independent of the class and the signal, and the two values are independent given the class and the signal. The cost is independent of the class, both values and the signal. A stronger incumbent then can only lower entry, for every such experiment and every independent cost distribution

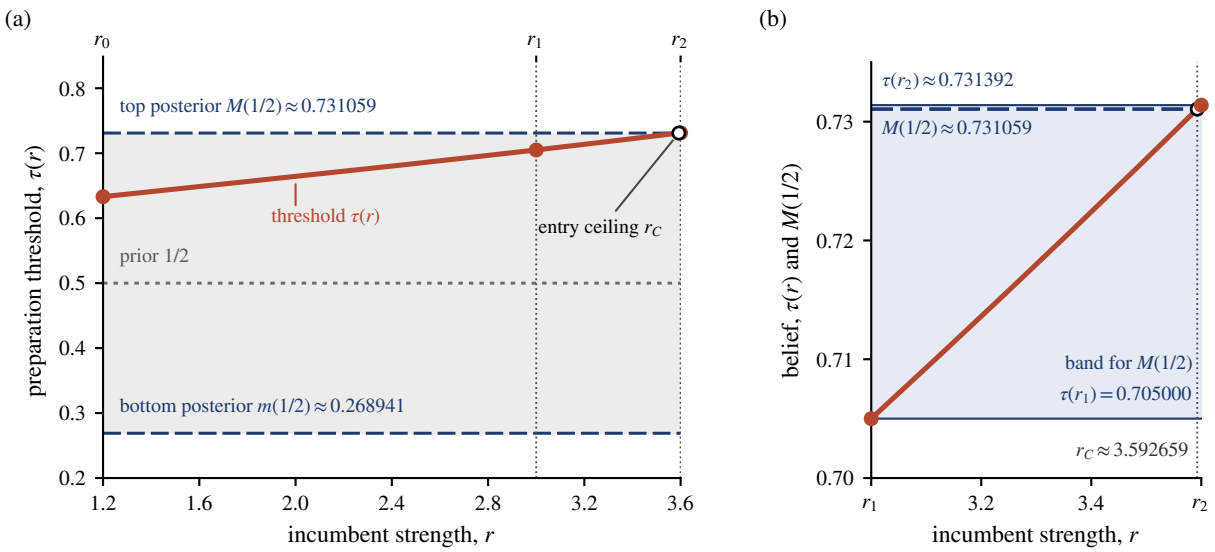


Figure 1. Beliefs and the entry ceiling in the example.

Notes. Panel (a) plots the preparation threshold $\tau(r)$ against incumbent strength r. The gray band covers the posteriors that a strategy of the informed trader can produce, from $m(1/2)$ to $M(1/2)$. The entry ceiling r_C is the strength at which $\tau(r)$ equals $M(1/2)$. Panel (b) enlarges the strengths from r_1 to r_2. Its blue band holds every top posterior $M(1/2)$ that puts r_C between r_1 and r_2; the example's value lies 0.000334 below the upper edge $\tau(r_2)$.

(Proposition A.2 in Appendix A.6 gives the formal condition): it lowers gross profit at every belief, so it can only remove a preparation incentive, and entry falls by exactly the probability that the cost lies between gross profit against the two incumbent distributions.

In the order slot the experiment fixes λ, β, π, the cost distribution and both order distributions, and competitive prices then reveal the same posterior after each order at every strength (Appendix A.2).

Panel B of Table 3 holds full orders fixed at both strengths. Only the top posterior can recruit a high-cost rival, because the preparation threshold $\tau(r)$ exceeds the prior at every strength, so entry is 0.522757 at every strength up to the entry ceiling $r_C \approx 3.592659$ and ρ above it. Entry falls by the probability that the cost lies between gross profit at the two strengths, which is 0 from r_0 to r_1 and 0.272757 from r_1 to r_2 (Proposition A.2): with information fixed, entry stays constant from r_0 to r_1 and falls above the entry ceiling.

Panel C hides the price from the rival, which then decides on its prior and prepares only at zero cost, so entry stays at 0.250000 at both strengths even though the informed trader trades fully at r_1. The two controls together locate the reversal in the information that trading generates: incumbent strength changes that information, and the entry response to it exceeds the direct deterrence effect of a stronger incumbent at fixed information.

Table 3. Equilibrium outcomes and information controls in the example.

	$r_0 = 6/5$	$r_1 = 3$	$r_2 = 18/5$
Panel A. Equilibrium (Proposition 2)			
Informed orders, classes H and L	$(0, 0)$	$(1, -1)$	$(1, -1)$
Posterior after a sale	0.500000	0.268941	0.268941
Posterior after no order	0.500000	0.500000	0.500000
Posterior after a purchase	0.500000	0.731059	0.731059
High-cost preparation	none	after a purchase	none
Price after a sale	0.392708	0.576074	0.610002
Price after no order	0.392708	0.614583	0.664236
Price after a purchase	0.392708	1.362372	0.718471
Entry E	0.250000	0.522757	0.250000
High-class ownership O_H	0.125000	0.324401	0.125000
Expected target proceeds	0.392708	0.872531	0.664236
Panel B. Full orders held fixed (Proposition A.2)			
Entry E	0.522757	0.522757	
High-class ownership O_H	0.324401	0.324401	
Expected target proceeds	0.503998	0.872531	
Panel C. The rival does not see the price			
Informed orders, classes H and L	$(0, 0)$	$(1, -1)$	
Entry E	0.250000	0.250000	
High-class ownership O_H	0.125000	0.125000	
Expected target proceeds	0.392708	0.614583	

Notes. The example. Panel A: the unique equilibrium at each strength. Panel B: full orders imposed at both strengths; at r_0 this profile is a control, not an equilibrium. Panel C: the unique equilibrium when the rival decides on its prior. Panels B and C compare r_0 with r_1; at r_2 the full-order profile is the equilibrium of Panel A.

4.3 Coexistence

Between the weak and the strong incumbent of Proposition 2, two equilibria can hold against the same incumbent. In an uninformative market the informed trader does not trade, the price carries no information, and entry stays at its floor; in the informative market that coexists with it, a purchase recruits a high-cost rival. The two numbers of Section 3.4 decide where both exist: the threshold must lie above the prior and within reach of a purchase, and the spread must be wide enough that a purchase that just reaches the threshold covers the trading cost, yet narrow enough that no order against an uninformative price earns more than it.

Proposition 3 (Uninformative and informative markets)**.** *Let the classes satisfy the class ranking* (1)*, let* $0 < \pi < 1$*,* $0 < \rho < 1$*,* $k > 0$ *and a slot* (λ, β) *be given, and let an indifferent informed trader mix over its best orders and an indifferent high-cost rival prepare. Take an incumbent distribution with a positive spread* Δ_T *and* $g_H > g_L$*, and write* τ *for its preparation threshold.*

(a) An uninformative market and an informative market coexist if and only if

$$\pi < \tau \leq M(\pi), \qquad \frac{k}{1-\tau} \leq \Delta_T \leq \frac{k}{\rho \max\{\pi, 1-\pi\}}.$$

The informative market is then unique: class L abstains, only a purchase recruits a high-cost rival, and entry is

$$\rho + (1-\rho) \min\left\{\frac{(1-\pi)\, b\, \Delta_T}{k},\ b + \pi\lambda\right\},$$

which rises strictly with the spread while class H mixes between a purchase and abstention and is constant once class H buys for sure.

(b) Along a path of such distributions, the strengths at which an uninformative market coexists with an informative market in which class H buys for sure form an interval, and every other strength with coexistence lies below it. Every gap in coexistence lies at spreads too narrow for a sure purchase to pay.

(c) Along a path on which the product of the spread and $g_H - c_H$ *does not fall with strength, the strengths with coexistence form an interval, at every prior. Uniform incumbents on* $[0, r]$*, with* r *in a closed interval above the reserve, satisfy this condition when every low-class value lies between the reserve and the lowest strength, every high-class value is at least the highest strength, and class H's mean value exceeds the highest strength by at least* c_H*.*

Coexistence comes from the entry response itself. In an uninformative market entry stays at its floor after every order, so an informed order earns at most $\rho \max\{\pi, 1-\pi\}$ times the spread, and the upper bound on the spread keeps that from exceeding the trading cost. An informative market

changes the entry response to a purchase, and with it the return to trade: a purchase that reaches the threshold recruits, and the lower bound on the spread lets it cover the trading cost.

Entry rises with the spread while class H mixes, because the purchase rate adjusts. While class H mixes, a recruiting purchase earns exactly the trading cost, so a wider spread needs a more informative purchase; the purchase rate rises, and entry rises with it until class H buys for sure. Entry depends on the incumbent only through the spread, so between two incumbents at which the markets coexist, entry in the informative market is at least as high against the one with the wider spread.

A stronger incumbent never narrows the spread, and where the threshold is positive it never lowers the threshold, so the two numbers pull in opposite directions: a wider spread raises what a recruiting purchase earns, and a higher threshold makes recruitment harder. As the incumbent strengthens, the economy can enter the region, leave it, and enter it again. The strengths with a sure purchase are never split, because each condition that defines them holds on one side of a single strength, and a mixed purchase needs a narrower spread than a sure one, so the other strengths with coexistence lie below them. When class H's gross profit exceeds the high cost against the weakest incumbent, the whole set stays in one piece if the spread grows, in proportion, at least as fast as that excess shrinks; part (c) states the condition as a product, which does not fall.

The example satisfies the condition of Proposition 3(c): its incumbent is uniform, ℓ lies between p and r_0, and h exceeds r_2 by more than c_H, so at every prior the strengths with coexistence form one interval. At the prior $1/2$ the interval starts at r_A, where a purchase that just reaches the threshold and recruits earns the trading cost, and it ends at r_N, where a unit order against an uninformative price earns the trading cost. Class H buys for sure from r_F, where a full purchase that recruits earns the trading cost.

Corollary 2 (Coexistence in the example)**.** *In the example at the prior* $1/2$*, let the slot lie in the band of Corollary 1, and let an indifferent informed trader mix over its best orders and an indifferent high-cost rival prepare. An equilibrium exists at every strength of the path from* r_0 *to* r_2*, and the strengths at which an uninformative market and an informative market coexist form the interval* $[r_A, r_N]$*. In the informative market that coexists with an uninformative market, entry is* $\rho + (1-\rho)\min\{b\Delta_T(r)/(2k),\ b + \lambda/2\}$*; it rises strictly on* $[r_A, r_F]$ *and is constant on* $[r_F, r_N]$*. With a full purchase, entry keeps this value up to the entry ceiling* r_C *and equals* ρ *above it.*

In the example's slot, coexistence runs from $r_A \approx 1.393958$ to $r_N \approx 1.747878$, and class H buys for sure from $r_F \approx 1.467127$. Between r_A and r_F class H buys at the rate $b\,[\Delta_T(r)/k - 2]$, which rises with the spread until it reaches λ at r_F, so entry in the informative market is 0.454185 at r_A, 0.459588 at 1.40 and 0.506112 at 1.45; from r_F it stays at 0.522757 up to the entry ceiling $r_C \approx 3.592659$. Above r_N the informative market with a full purchase remains, and on (r_N, r_J),

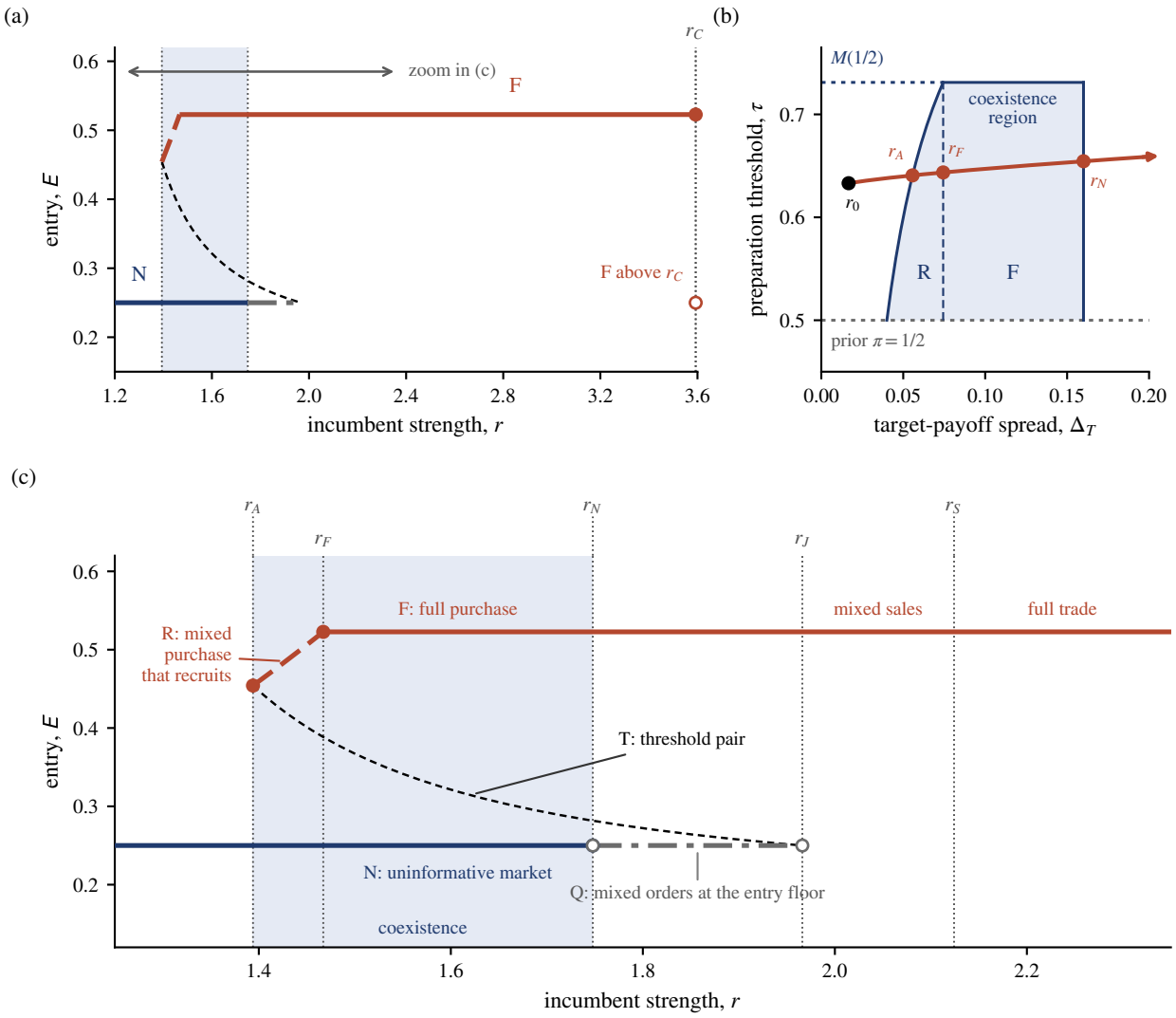


Figure 2. Entry and the coexistence region along the example's path.

Notes. The example at the prior $1/2$, when an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares. Panels (a) and (c) show entry against incumbent strength r in each equilibrium of Corollary 2. Panel (a) covers the path from r_0 to r_2, and panel (c) enlarges the strengths around the coexistence interval, which the blue band marks from r_A to r_N. The figure labels an uninformative market by N, a mixed purchase that recruits by R, a full purchase by F, mixed orders at the entry floor by Q and the threshold pair by T. The thin dashed line is the threshold pair T. At r_A its purchase recruits with probability one, and it is an equilibrium under the default tie rules; on $(r_A, r_J]$ it is an equilibrium when an indifferent high-cost rival may prepare with any probability. With a full purchase the low class abstains up to r_N, mixes from r_N to r_S, and sells with probability one from r_S; above the entry ceiling r_C, entry with a full purchase equals ρ. Panel (b) draws the region of Proposition 3(a) at the example's slot, in the plane of the target-payoff spread Δ_T and the preparation threshold τ: $1/2 < \tau \le M(1/2)$ and $k/(1-\tau) \le \Delta_T \le 2k/\rho$. The dotted lower edge, where τ equals the prior, lies outside the region, and the other three edges belong to it. The dashed line $\Delta_T = k/(1 - M(1/2))$ divides the mixed purchase R, on its left, from the full purchase F. The curve traces $(\Delta_T(r), \tau(r))$ from r_0: it enters the region at r_A, reaches full purchases at r_F, leaves the region at r_N, and continues towards r_1 and r_2 beyond the panel. Filled markers belong to a curve, and open markers do not.

with $r_J \approx 1.966097$, mixed orders at the entry floor join it. The low class abstains up to r_N, mixes on (r_N, r_S) and sells for sure from $r_S \approx 2.124148$, the strength at which a full sale earns the trading cost. Figure 2 draws the path through the region and the entry it produces, and Appendix A.4 lists every equilibrium along the path.

The example's high cost lies between the two bounds that these conditions place on it. It is at most $h - r_2 = 32/5$, so class H's value covers the high cost against the strongest incumbent and Proposition 3(c) applies, and it is at least $(h^2 - p^2)/(2h) = 399/80$, so class H has the decisive advantage of Section 4.1.

Coexistence also holds away from the even prior. In the example's slot it starts at the same strength r_A for every prior between ≈ 0.396175 and ≈ 0.640739, and it starts later, where the prior reaches the threshold, for priors up to ≈ 0.650059. Below ≈ 0.396175 even the top posterior cannot reach the threshold once a purchase can pay for itself, and from ≈ 0.650059 on, the uninformative market ends before the prior falls below the threshold. Appendix A.4 gives the coexistence set at every prior (Corollary A.2).

5 Competition creates research

With research, the informed trader chooses the presence probability λ of Section 2.3 and pays for it. Before it learns the class, the informed trader chooses a research intensity $\lambda \in [0, \bar{\lambda}]$ at the cost $(a/2)\lambda^2$, where $a > 0$ and the cap satisfies $0 < \bar{\lambda} < 1$. Research succeeds with probability λ, independently of the class, the costs and the incumbent's value. A success reveals the class, and the informed trader then fills the slot with an order. After a failure, a liquidity trader fills the slot.

Research is therefore the presence probability of Section 2.3, chosen at a cost. It is the informed trader's own work on the target's customers, technology and demand, and it must end before the order opportunity. The cap $\bar{\lambda}$ is the probability that this work can reveal the class at all before the rival decides, because some facts about the match become visible only through the rival's own diligence. The marginal cost rises because the informed trader uses cheap sources first, such as its sector coverage, and costly ones later, such as expert calls. The quadratic cost is the simplest form whose marginal cost rises.

A success takes the place of a liquidity order because the opportunity has room for one order. An informed trader that knows the class takes it. After a failure, the informed trader knows no more than the market, so an order would lose at least its cost k. The opportunity then passes to a liquidity trader. More research therefore also removes liquidity orders, and it reduces the camouflage of the informed trader's own orders.

Market makers and the rival do not observe the research choice, so they price and decide at a conjectured intensity. The prior is $1/2$ throughout. In an equilibrium with research, the conjectured

intensity is the informed trader's best research choice at the equilibrium prices and entry rule, and the order after a success is optimal, including after a deviation from zero research. The best research choice is unique: it equals the expected trading payoff after a success divided by a, capped at $\bar{\lambda}$, so research never has a tie. Appendix A.5 states the equilibrium with research formally and gives each result on research in full.

Research raises the top posterior that a purchase can reach. At intensity λ, I write M_λ for that top posterior and $m_\lambda = 1 - M_\lambda$ for the bottom posterior, the posterior bounds of Section 3.2 at the prior $1/2$ when the presence probability is λ. When $1/2 < \tau < 1$, the *recruitment intensity* $\lambda_E(\tau)$ is the research level at which a full purchase just reaches the threshold, $M_\lambda = \tau$, so a full purchase recruits exactly when research is at least $\lambda_E(\tau)$.

Consider the equilibrium in which class H buys after a success and a purchase recruits a high-cost rival. When an indifferent informed trader may mix over its best orders, class L abstains while a sale at the entry floor cannot pay its cost, $\rho\Delta_T \le 2k$; beyond that point it sells just often enough that its sale breaks even, and once even a sure sale pays, it sells for sure. A sale never recruits, and a sale that breaks even adds nothing to the value of research, so research optimality at an interior intensity reads

$$2a\lambda = \Delta_T\, m_\lambda - k + \big(\rho\Delta_T\, m_\lambda - k\big)^+. \tag{10}$$

The left side rises with λ, and the right side falls, because more research removes camouflage. When $\Delta_T > 2k$, the two sides meet at one intensity $\lambda_R(\Delta_T) \in (0, 1)$, and research is the smaller of $\lambda_R(\Delta_T)$ and the cap $\bar{\lambda}$. The *exact recruitment margin* is the right side of (10) minus its left side at the recruitment intensity; it is positive exactly when, at the research level that just lets a purchase recruit, the value of a success exceeds the marginal cost of research.

Proposition 4 (Competition creates research)**.** *Let the classes satisfy the class ranking* (1) *and the prior be* $1/2$*, and let an indifferent informed trader mix over its best orders and an indifferent high-cost rival prepare. Take incumbent distributions with a positive spread,* $g_H > g_L$ *and* $\tau > 1/2$*, and consider the equilibrium in which class H buys after a success and a purchase recruits a high-cost rival. The same recruiting outcome is available when an indifferent high-cost rival may prepare with any probability and prepares fully at the purchase.*

(a) The equilibrium exists if and only if $\Delta_T > 2k$ *and* $\tau \le M_\lambda$ *at* $\lambda = \min\{\lambda_R(\Delta_T), \bar{\lambda}\}$*, which is then the informed trader's research.*

(b) Research rises strictly with the spread until it reaches the cap and then stays at the cap. Entry, $\rho + (1-\rho)[\beta + (1/2 - \beta)\lambda]$*, and entry in class H rise with research, and entry in class L falls.*

(c) When class H outbids the incumbent, high-class ownership equals half the entry in class H. Along a path on which class H outbids every incumbent, ownership does not fall with strength,

and it rises strictly where research is below the cap and the spread rises strictly.

(d) Along a compact path of such distributions with continuous spread and threshold, let the spread against
the first strength at which the spread reaches that value, and let the threshold stay below one up to $r_{\bar{\lambda}}$. If the exact recruitment margin rises strictly up to $r_{\bar{\lambda}}$, from a negative value against the weakest incumbent to a positive value at $r_{\bar{\lambda}}$, then the equilibrium exists exactly on an interval of strengths. On it, research rises from the recruitment intensity to the cap and then stays at the cap, up to the last strength at which the threshold does not exceed the top posterior at the cap.

A wider spread raises the return to a purchase, so the informed trader chooses more research, and a purchase becomes more frequent, and its posterior rises. Research adds informed purchases in class H and removes liquidity orders in both classes, and the first effect dominates, so entry rises with slope $(1-\rho)(1/2-\beta)$ in research. Entry in class H rises and entry in class L falls, so the rivals who prepare are more often of class H.

The low class's selling leaves this mechanism in place. A sale never recruits, and while the low class mixes, its sale breaks even, so a success has the same value as when the low class abstains. Once a sure sale pays, the sale's profit is a second return to research, and if research is still below its cap at that point, the research schedule kinks upward there.

High-class ownership multiplies entry in class H by class H's chance of beating the incumbent. When class H outbids the incumbent, that chance is one, so ownership rises with research, as part (c) states. A decisive advantage of class H (Section 4.1) supplies this outbidding at every incumbent against which a purchase recruits.

Along a path the spread and the threshold race as in Section 4.3: a stronger incumbent raises what a recruiting purchase earns and the belief it must reach. When the exact recruitment margin rises up to $r_{\bar{\lambda}}$, the race tips toward recruitment once below the cap, and part (d) keeps the equilibrium on one interval of strengths, which persists through the last strength of the path at which the threshold does not exceed the top posterior at the cap. Under the cheap-research condition, the increase in a recruiting purchase's return exceeds the marginal cost of the additional research needed to reach the threshold, so the exact recruitment margin rises. Appendix A.5 states the condition on these slopes and gives the full statement and proof of Proposition 4.

Corollary 3 (Research equilibria in the example)**.** *In the example at the prior $1/2$, with the research cost $a = 1/40$ and the cap $\bar{\lambda} \approx 0.336122$, let an indifferent informed trader mix over its best orders and an indifferent high-cost rival prepare. Write $r_R \approx 1.501023$, $r_{\bar{\lambda}} \approx 1.677633$, $r_N \approx 1.747878$, $r_Q \approx 2.166995$ and $r_C \approx 3.592659$. On the path from r_0 to r_2:*

(i) zero research, with entry ρ, is an equilibrium exactly on $[r_0, r_N]$;

Table 4. Research and entry at the intermediate strengths.

	$r = 1.55$	$r = 1.60$	$r = 1.65$
Target-payoff spread $\Delta_T(r)$	0.097581	0.112500	0.128030
Preparation threshold $\tau(r)$	0.646798	0.648734	0.650670
Recruitment intensity $\lambda_E(r)$	0.196741	0.199698	0.202664
Research $\lambda_R(r)$	0.237901	0.279037	0.316683
Posterior after a purchase M_{λ_R}	0.673142	0.698206	0.720112
Entry on the recruiting branch	0.507630	0.513965	0.519763
High-class ownership on the recruiting branch	0.298422	0.309302	0.319260
Entry without research	0.250000	0.250000	0.250000

Notes. The example with $a = 1/40$ and $\bar{\lambda} \approx 0.336122$, with $\lambda_E(r) = \lambda_E(\tau(r))$ and $\lambda_R(r) = \lambda_R(\Delta_T(r))$. The spread, the threshold and λ_E are rational numbers; the other entries are rounded to six places.

(ii) recruiting research is an equilibrium exactly on $[r_R, r_C]$*; research is below the cap on* $[r_R, r_{\bar{\lambda}})$ *and at the cap on* $[r_{\bar{\lambda}}, r_C]$*, and entry rises from* 0.500846 *at* r_R *to* 0.522757 *at* $r_{\bar{\lambda}}$ *and keeps that value up to* r_C*;*

(iii) floor-entry research, in which both classes trade after a success, no order recruits and entry is ρ*, is an equilibrium exactly on* (r_N, r_Q)*, below the cap, and on* $(r_C, r_2]$*, at the cap;*

(iv) there is no other equilibrium.

The unique equilibrium thus has zero research at r_0*, recruiting research at the cap at* r_1 *and floor-entry research at the cap at* r_2*; at* r_1 *and* r_2 *it is the equilibrium of Proposition 2 with* $\lambda = \bar{\lambda}$*.*

Appendix A.5 states Corollary 3 in full, with the low class's orders along each equilibrium and the equilibria at the three intermediate strengths of Table 4, and proves it.

Research and entry rise together across the three intermediate strengths in Table 4: research rises from 0.237901 through 0.279037 to 0.316683, and entry rises from 0.507630 through 0.513965 to 0.519763. A stronger incumbent widens the target-payoff spread, so a purchase earns more and the informed trader chooses more research; the posterior after a purchase then rises above the threshold by more, and a purchase becomes more frequent. Proposition 4(b) states the same comovement wherever the equilibrium exists.

The research cap is the presence probability of the example's slot, so research at the cap reproduces that slot. The example meets the condition of Proposition 4(d): the exact recruitment margin rises strictly from r_0 to $r_{\bar{\lambda}}$, is negative at r_0 and is positive at $r_{\bar{\lambda}}$. A researched purchase first recruits at r_R, where a purchase that just reaches the threshold also pays for its research, $\Delta_T(1-\tau) = k + 2a\lambda_E$; there research is 0.193853 and entry 0.500846. Research reaches its cap at $r_{\bar{\lambda}}$ and stays there up to the entry ceiling. The low class starts to sell at r_N and sells for sure from

$r_S \approx 2.124148$; research is at its cap before either strength, so in the example the sale's profit leaves research unchanged.

Zero research is an equilibrium on $[r_0, r_N]$, and an informed order at the entry floor starts to pay just above r_N. Floor-entry research coexists with recruiting research exactly on (r_N, r_Q). Recruiting research coexists with zero research on $[r_R, r_N]$, is the only equilibrium on $[r_Q, r_C]$, and gives way to floor-entry research at the cap on $(r_C, r_2]$. Against the weak incumbent every nonzero order after a success loses, so research is zero there. Figure 3 draws the three equilibria along the path.

The recruiting branch in Figure 3 starts later than the informative branch of Section 4.3, which starts at $r_A \approx 1.393958$, because a purchase must pay for research as well as for trade. The two branches also adjust in different ways: with a fixed presence probability the purchase rate adjusts through a mixture of the high class, while with research the intensity adjusts and the high class buys with probability one after a success.

Proposition A.1 in Appendix A.5 gives the research parameter region exactly at $\beta = 0.294655$. The pattern of Corollary 3 at these strengths (zero research alone at r_0, zero research and recruiting research below the cap at each of the three intermediate strengths, and a unique outcome with full orders and research at the cap at r_1 and r_2) holds with strict margins exactly when the cap lies in the strip $(\lambda_E(r_1), \lambda_E(r_2)) \approx (0.290539, 0.336721)$ and the cost lies in a window that depends on the cap. At the example's cap, this window is $(0.021469, 0.036763)$. With $1 + u = \mathrm{e}$, the cap $\bar{\lambda} = u\beta/(1 + u\beta)$ and $a = 1/40$, this pattern holds for every $\beta \in (\beta_{\mathrm{lo}}, 1/3]$, with $\beta_{\mathrm{lo}} \approx 0.236157$. Entry at r_1 is $\rho + (1 - \rho)[\beta + (1/2 - \beta)\bar{\lambda}]$.

6 Robustness

I vary one part of the model at a time: the preparation-cost distribution, the value classes and the prior, and the rival's information. Appendix A.6 states each result in full and proves it.

6.1 Preparation costs

The rise of entry in Proposition 2 survives other preparation-cost distributions, with or without atoms (Proposition A.3). The proposition lets the cost mix a low component with weight ρ and a high component in place of the two-point distribution of Section 2.2; either component can be atomless or have atoms. It keeps parts (i) and (ii) of Proposition 2 under the trading-cost window (H2) when every cost in the low component lies strictly below $B_{F_1}(m(\pi))$ and every cost in the high component lies strictly between $B_{F_0}(\pi)$ and $B_{F_1}(M(\pi))$. The low component then prepares at every feasible belief, so entry stays at least ρ, and the trading bounds of Section 3.3 apply unchanged.

In the example, the low component runs from zero to $1/5$, and the high component is uniform

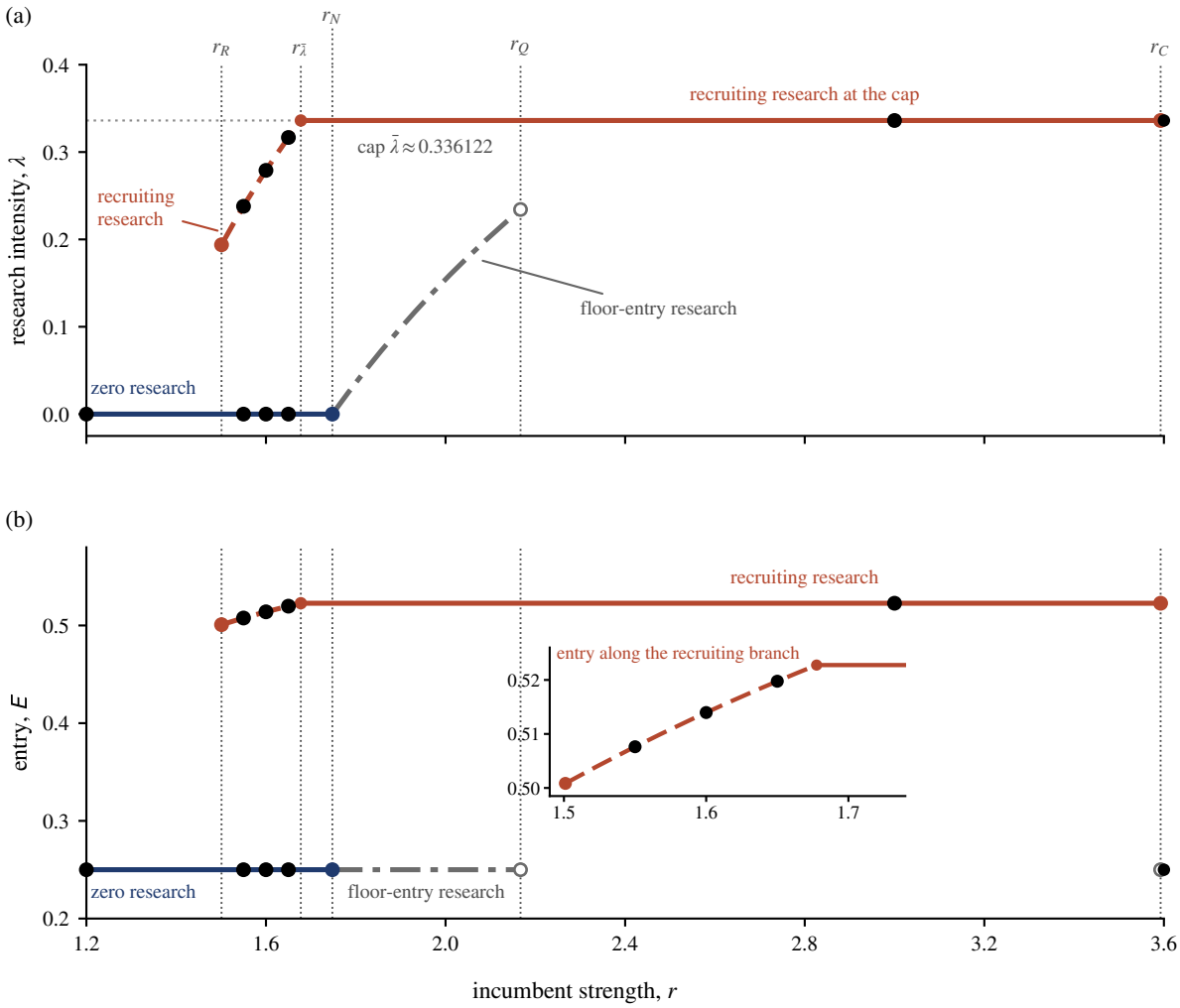


Figure 3. Research and entry along the example's path.

Notes. The example with $a = 1/40$ and $\bar\lambda \approx 0.336122$, at the prior $1/2$, when an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares. Panel (a) shows research λ and panel (b) shows entry; the inset enlarges entry along the recruiting branch. Zero research is an equilibrium from r_0 to r_N. Recruiting research starts at r_R, reaches the cap at $r_{\bar\lambda}$ and stays at the cap up to the entry ceiling r_C; its entry rises from 0.500846 to 0.522757 and keeps that value up to r_C. In recruiting research the low class abstains up to r_N, mixes from r_N to r_S, and sells with probability one from r_S. On (r_N, r_Q), floor-entry research coexists with it: both classes trade after a success, no order recruits, entry stays at ρ, and research rises towards 0.234257 as strength approaches r_Q from below. From r_Q to r_C, recruiting research at the cap is the only equilibrium. Above r_C, floor-entry research at the cap is the only equilibrium, with entry ρ. Black dots mark r_0, the three intermediate strengths, r_1 and r_2. Filled markers belong to a curve, and open markers do not; at r_N the filled marker belongs to zero research, and at r_C to recruiting research at the cap.

on $[c_H - \varepsilon, c_H + \varepsilon]$. Every half-width below $c_H - B_{F_2}(M(1/2)) \approx 0.002689$ keeps all three parts of Proposition 2. With half-width $\varepsilon = 1/10$, entry at r_1 is 0.522757, the value under the two-point distribution. At r_2, gross profit at the top posterior falls inside this band, and entry at r_2 is 0.382711. With an atomless high component, the drop at the entry ceiling becomes a ramp: full-order entry is continuous and nonincreasing in strength.

6.2 Value classes and priors

The reversal holds for every pair of ranked classes and every prior. The classes satisfy the class ranking (1), class H has the higher mean value, and the reserve, the prior, the zero-cost probability and the slot are arbitrary. For uniform incumbents, a nonempty open set of strengths and costs then satisfies (H1)–(H3) and the ownership condition of Proposition 2 (Proposition A.4).

The construction puts the weak and the strong incumbent just above the lowest value θ_* at which the classes start to differ above the reserve. The spread vanishes near θ_*, so informed trade against the weaker incumbent cannot cover its cost. The profit gap stays near the full class gap there, so a high cost that only a favorable price justifies fits inside it. In the point-mass case $\theta_* = \ell$, so the reversal holds for every $h > \ell$.

Table A.4 in Appendix A.6 reports six economies with their own costs and strengths. With wide uniform classes, entry rises from 0.250000 to 0.573175, and high-class ownership rises although the high class can lose to the strong incumbent; point masses at the same means leave entry at 0.250000. A prepared rival keeps $(\theta - \max\{p, R\})_+$, a convex function of its value, so a mean-preserving increase in risk within a class never lowers gross profit at any belief.

With more than two classes, the reversal holds under a separation condition (Proposition A.5). The price reveals only the posterior mean of class proceeds. Separation splits the classes into a high group and a low group, and it asks for a gap between the proceeds of each high-group class and every feasible posterior mean, and the same gap for the low group. In the strong equilibrium, class proportions within each group then stay at their prior proportions, so the price reveals the whole posterior vector.

A four-class economy near the example satisfies separation with distance 0.178871. Without separation, one price can carry two posterior vectors: a strict three-class equilibrium has two orders with different posterior vectors and one price.

6.3 Complementary private signals

The rival can gain from the price even when its own information is more accurate than the informed trader's. A rival knows its integration plans, while informed traders who follow the target know its customers or its product market. Proposition A.6 represents the two sources with conditionally

independent binary signals. It holds at the prior $1/2$ for every pair of ranked classes and every weak and strong incumbent with continuous distributions. The informed trader's signal has accuracy α_I, and the rival's signal has accuracy α_C.

The zero-cost rival still prepares after every price and signal, so the price still reveals the public posterior, and the rival combines that posterior with its own signal. Under strict conditions on the costs, the weak incumbent has zero orders and entry ρ; the strong incumbent has full orders and entry above ρ. The conditions allow $\alpha_C > \alpha_I$.

The signal example uses its own parameter vector. It keeps the slot, the values, the reserve and the prior of the example, and it sets $\rho = 0.85$, $c_H = 7.14$, $k = 0.015$, $r_0 = 1.1$ and $r_1 = 2.3$. With accuracy $\alpha_I = 0.70$ for the informed trader's signal and $\alpha_C = 0.75$ for the rival's signal, entry rises from 0.850000 to 0.879797. A favorable signal alone does not justify high-cost preparation against the weak incumbent; a favorable signal and a favorable price together do so against the strong incumbent. Both sources are needed: without the rival's signal, entry is 0.850000 at both strengths.

In the example itself, the conditions hold with $(\alpha_I, \alpha_C) = (0.94, 0.505)$. Entry then rises from 0.250000 to 0.386933, and without the rival's signal it is 0.250000 at both strengths. In the example the conditions cannot hold when the rival's signal is at least as accurate as the informed trader's: the strong trading bound needs an accurate signal for the informed trader, and the weak bound then fails. The signal example meets the conditions with a more accurate signal for the rival at other costs and strengths.

7 Welfare and sale terms

7.1 Access to prices

Access to prices improves acquisition outcomes at a fixed level of competition. At the strong incumbent F_1, I compare the equilibrium of Proposition 2 with that of an economy in which the rival does not see the price. Under the hypotheses of Proposition 2, both economies have unique full orders, and price access raises expected target proceeds and acquisition surplus net of preparation costs (Proposition A.7). Acquisition surplus excludes transfers between bidders, shareholders and traders, and Appendix A.7 states each result on welfare and sale terms in full.

The two economies generate the same orders. The trading bound of Section 3.3 forces full orders even when the rival cannot use the price and entry stays at ρ, so trading costs coincide. Price access changes only which high-cost rivals prepare: it adds preparation after a purchase. Each added preparation adds its gross profit minus its cost to net surplus, together with the value of any sale that the reserve would otherwise block; a rival prepares only when its expected gross profit covers its cost, so no added preparation lowers net surplus.

In the example at r_1, expected proceeds rise from 0.614583 to 0.872531, a gain of 0.257948, and net surplus rises by 0.081960. With research at the cap (Section 5), the gains are the same.

Target shareholders gain because a shareholder who holds through the sale receives V_T, not the interim price. The informed trader's expected gross profit, 0.037665 at r_1, is a transfer from liquidity traders. If one order unit is a given number of shares per share of float, initial shareholders gain in total relative to an economy without the informed trader whenever that number is below 6.848394, even if they bear the whole transfer.

Inside the recruitment region of Proposition 2, where high-cost preparation does not pay at the prior, price access adds preparation. When gross profit at the prior exceeds c_H and gross profit after a sale falls below it, an informative price deters high-cost preparation after a sale: in the example at r_1 with $c_H = 12/5$, price access changes proceeds by -0.173917 and net surplus by -0.013505 (Appendix A.7).

The two economies differ in the information in the price and not in its level. A deterministic external dividend of 0.257948 in the hidden-price economy equalizes the mean prices, and it leaves the orders and entry unchanged.

7.2 The payment rule

The effect of competition on the target-payoff spread depends on the sale payment. Proposition A.8 isolates this dependence at the payment stage, in a verifiable-value institution with zero reserve, point-mass classes and an incumbent distribution on $[0, \bar{r}]$ with $0 < \ell < \bar{r} < h$. After diligence, a sale to the next bidder at its value is an enforceable fallback, and the seller and the winner divide the surplus above the fallback by Nash bargaining, with seller weight $\omega \in [0, 1)$. A stronger incumbent never raises the rival's profit, and it moves the spread only in the direction of the sign of $1 - 2\omega$; the move is strict when $\mathbb{E}[(R - \ell)_+]$ changes strictly and $1 - 2\omega \neq 0$, and when $1 - 2\omega = 0$ the spread equals $(h - \ell)/2$ at every incumbent distribution.

To see why, take $R > \ell$. A high-class rival wins and pays $(1 - \omega)R + \omega h$, whose sensitivity to R is $1 - \omega$. A low-class rival loses, and the incumbent pays $(1 - \omega)\ell + \omega R$, whose sensitivity is ω. Competition widens the difference exactly when the first sensitivity exceeds the second.

In the example, the spread under this rule changes by $13/20\,(1 - 2\omega)$ from r_0 to r_1. The second-price auction is the case $\omega = 0$: it puts the whole weight on the runner-up's value. **?** compare an auction with one more bidder and an optimal negotiation by expected revenue; here the comparison concerns how competition moves the spread.

Table 5. Reserves, preparation and sale in the example.

Reserve	Strength	Entry	Sale	Two admissible bidders	Expected proceeds
$1/2$	r_0	0.250000	0.687500	0.145833	0.392708
	r_1	0.522757	0.920459	0.435631	0.872531
$101/100$	r_0	0.522757	0.431371	0.051364	0.440564
	r_1	0.522757	0.772548	0.215186	0.994384
$31/20$	r_0	0.522757	0.324401	0	0.502822
	r_1	0.250000	0.547917	0.060417	0.893073

Notes. The example, with point-mass classes. At each reserve, full informed orders are the unique outcome at both strengths, except at the reserve $1/2$ and r_0, where the informed trader does not trade.

7.3 Reserves

A reserve affects both the payment required to win and the information in the target's shares. Once the reserve can exclude bidders, preparation, a sale and the meeting of two admissible bidders become different events: *preparation* is the event that the rival pays its preparation cost, a *sale* occurs when at least one bidder submits an admissible bid, and *two admissible bidders* meet when both the incumbent and a prepared rival meet the reserve. In the example of Section 4.2, the reserve lies below both class values of the rival, so every prepared rival is admissible.

A reserve that excludes the low class can widen the class difference in target proceeds, because the low class then adds nothing to proceeds. Informed trade can then pay against both incumbents, and the price is informative at both strengths. Proposition A.9 gives the consequence. If informed trade is full at both strengths and only a purchase recruits at each, entry is flat in incumbent strength, and strict deterrence needs the stronger incumbent to push gross profit at the top posterior below the high cost.

Table 5 reports three reserves in the example. The reserve $101/100$ excludes the low class. Full orders are unique at both strengths, only a purchase recruits, and entry is 0.522757 at both r_0 and r_1, while this reserve raises expected proceeds at both strengths. Narrow uniform classes give the same pattern at the reserve $11/10$.

The reserve $31/20$ restores strict deterrence: entry is 0.522757 at r_0 and 0.250000 at r_1, and proceeds exceed their levels at the reserve $1/2$. This reserve excludes the low class and also the weak incumbent, since it exceeds r_0. In the example, strict deterrence with a reserve that excludes the low class requires the reserve to exclude the weak incumbent as well. With a reserve in $(\ell, r_0]$, the top posterior recruits at both strengths, and entry is flat. Strict deterrence holds for reserves between about 1.325270 and 1.792723 (Appendix A.7).

8 Empirical implications

The model gives predictions for publicly visible sale processes in which one bidder is prepared and a rival has not yet committed to diligence. Each prediction names the incumbent, prior, trading-pattern and tie-rule conditions it needs, and concerns one target in one public state; Appendix A.8 states each in full and proves it. The predictions concern price impacts, the informed trader's presence, the winner and announcement returns. Under full two-sided trade, class H buys and class L sells, each with probability one, and the informed share of nonzero orders is $u/(2+u)$ at every prior. In the example at the prior $1/2$ and the example's slot, full two-sided trade is the unique outcome at every strength between r_S, the strength at which a full sale earns the trading cost, and h; this range contains r_1, the entry ceiling r_C and r_2.

Entry and incumbent strength. When the price reaches the rival before its decision, a stronger incumbent can raise the probability that the rival prepares; in the example, that probability rises by 0.272757 from r_0 to r_1. When the price does not reach the rival, a stronger incumbent can only lower it. Against a strong enough incumbent, entry can return to its floor while the price stays informative. Proposition 2 gives the first and third comparisons, and the fixed-information control of Section 4.2 gives the second (Proposition A.2).

The price gap and its entry component. Under full two-sided trade at a prior below the preparation threshold, the purchase impact $P(1) - P(0)$, weighted by the frequency of a purchase, exceeds the sale impact $P(0) - P(-1)$, weighted by the frequency of a sale, exactly when a purchase recruits the rival (Proposition A.10). The weighted difference equals the expected price before the order opportunity minus the price after no order, so the test needs prices and order frequencies and no auction object; at the prior $1/2$ the two frequencies are equal, and the test compares the two impacts directly. Under the same conditions the price gap $P(1) - P(-1)$ has a class component, $\rho\Delta_T\,[M(\pi) - m(\pi)]$, and an entry component, which is positive exactly when a purchase recruits; the whole entry component sits in the purchase impact. As the incumbent strengthens, entry adds more to the price through competition and less by rescuing a sale that the incumbent alone would not complete. With a uniform incumbent and the prior and the slot fixed, the entry component rises with strength while full two-sided trade and recruitment persist, when $3p^2 < \min\{\underline{\theta}, \underline{r}\}^2$, where $\underline{\theta}$ is the lowest value of the rival and $\underline{r}$ the lowest strength compared; the example meets this condition.

In the example at r_1, the price gap is 0.786299, and its entry component is 90.2% of it. With a hidden price, the price gap would be 0.077020. At r_2 the price gap is 0.108469, the class component alone, and at the entry ceiling r_C it falls by 0.861835.

The informed share. Under full two-sided trade the mean order is $\lambda(2\pi-1)$, so order counts identify the informed trader's presence when the prior is known and differs from one half (Proposition A.11). The sale impact identifies presence without knowledge of the prior: when entry after a sale and after no order sits at its floor and the payoff scale $\rho\Delta_T$ is known, the sale impact weighted by the frequency of a sale, divided by $\rho\Delta_T$, equals $\pi(1-\pi)\lambda$, and together with the mean order $\lambda(2\pi-1)$ it identifies both the presence probability and the prior. At the prior $1/2$ the counts depend on the slot only through the volume of nonzero orders, so prices separate slots that counts cannot. In the example at r_1, the shares consistent with the counts and the informed trader's incentives form the interval from ≈ 0.250299 to 0.760000, and a slot with share 0.444444 has the example's volume and entry, but its price gap at r_1 is 0.778935, against 0.786299.

Winners. Under full two-sided trade, a prepared rival wins more often after a purchase than after a sale exactly when the incumbent's value falls, with positive probability, where a high-class value beats it with a higher probability than a low-class value; the gap is $M(\pi)-m(\pi)$ times the difference between the classes' chances of beating the incumbent (Proposition A.12). Orders predict the winner even where they do not predict entry: where entry stays at its floor after every order, as above the entry ceiling, the order still covaries positively with a win by the rival under the same condition, and not with entry. With the prior and the slot fixed and full two-sided trade at each strength compared, when class H outbids every incumbent the gap does not narrow as the incumbent strengthens, and it widens wherever the stronger incumbent takes wins from the low class. Among prepared rivals, the share of the high class at each price is the price-implied posterior, which rises with the price under full two-sided trade. In the example class H outbids every incumbent, and the gap is 0.308078 at r_1 and 0.333751 at r_2; the covariance of the order with entry is 0.272757 at r_1 and zero at r_2.

Announcements. For a positive target-payoff spread and $g_H > g_L$, if a public event after the price reveals only whether the rival prepared, the mean price change over preparation reports is positive exactly when preparation was still uncertain after some order, at every prior and in every equilibrium (Proposition A.13); in the example it is 0.160592 at r_1. The mean change over both reports, with their model probabilities, is zero. Under full two-sided trade at a prior below the preparation threshold, an opportunity that passes with no order is no news about the class and, when a purchase recruits, bad news about entry: in the example at r_1, the price after no order minus the expected price before the opportunity is -0.257948, and the size of this drop is the weighted difference of the price-impact test.

Two clusters at intermediate strengths. In the example, at one public strength in the range (r_F, r_N) of Section 4.3, processes can form two clusters. In one cluster the market is uninformative:

the mean order is zero and the three prices coincide. In the other, the high class buys and the low class abstains; the mean order is then $\lambda/2$ (0.168061 in the example), every rival prepares after a purchase, and a sale executes above the price after no order, by 0.005363 at 1.60.

Measurement. The counterpart of preparation is the start of substantive diligence or a proposal that requires costly preparation; public offers are a narrower outcome. Incumbent strength is measured from information that is public before that decision, and the price must precede it.

The order is the first prespecified public order opportunity after the sale becomes public, and an opportunity with no order is recorded as a zero. A scheduled call auction is one example, such as the opening auction on the first trading day after the approach is disclosed; its time is fixed in advance and does not depend on any trader's information, and the auction takes place whether or not an order arrives. The sign of its net order imbalance is the order, and an imbalance below a size fixed in advance records the zero order. A sample of first executed trades omits the zero order, so it cannot identify the volume, the effect of an inactive opportunity, or the informed share.

The announcement formulas need events that reveal only preparation. Target returns before a bid also partly reflect anticipation of that bid (**?**), so each test dates the price before the rival's decision. The trading counterpart is outside trade with a view on the rival's fit, such as orders from sector-focused funds.

9 Conclusion

In the model, the bidder pool in a takeover depends on the information that bidders have before they commit to preparation. A stronger incumbent moves the class value gap from the rival's profit to target proceeds, and because a liquidity order can match the informed trader's direction in the order slot, an informed trader who knows the rival's class keeps a residual advantage after every order. A larger target-payoff spread can make that advantage worth the cost of trade, so the price can recruit a rival that a weaker incumbent fails to attract.

The comparison of the weak, strong and very strong incumbent holds with unique outcomes on an open set of parameters (Proposition 2). Under the default tie rules, an uninformative market and an informative market coexist exactly on a region of the target-payoff spread and the preparation threshold (Proposition 3), and a stronger incumbent can move the economy into and out of that region; in the informative market, entry rises with the spread while the high class mixes between a purchase and abstention and stays constant once it buys for sure. When information is costly, competition creates research: at an even prior, when an indifferent informed trader may mix over its best orders and the preparation threshold lies above the prior, research and entry rise together with the spread until research reaches its cap, in the equilibrium in which a purchase after successful

research recruits (Proposition 4), and the low class's selling does not interrupt the rise.

Sale terms affect entry through the rival's expected payment and through the information in the stock price. Under the hypotheses of the main result, access to prices raises target proceeds and acquisition surplus net of preparation costs against the strong incumbent, and, with a single value in each class and the incumbent's highest value between them, in a bargaining family of payment rules with a zero reserve, the weight that the rule places on the runner-up's value decides the direction in which competition moves the target-payoff spread. A reserve that excludes the low class can make the price informative against both incumbents; if informed trade is then full and only a purchase recruits at both strengths, entry is flat in incumbent strength.

The next theoretical step is the seller's choice of terms with the whole pricing and trading continuation retained. The empirical tests ask whether the rival's preparation in visible sale processes responds to incumbent strength when the price precedes the decision, and, where the informed trader always buys after good news and always sells after bad news and the prior lies below the preparation threshold, whether a purchase raises the price by more than a sale lowers it once each impact is weighted by how often its order occurs, as Section 8 predicts where a purchase recruits.

References

A Proofs

This appendix proves every result in the paper. Statements use the class model of Section 2. A statement that needs the example says so. Each equilibrium proof states the equilibrium tuple of Definition A.1, lists its conditions, and shows that the fixed tuple satisfies each of them.

The equilibrium tuple. Write $\eta(-1) = \eta(1) = \beta$ and $\eta(0) = 1 - 2\beta$ for the liquidity order distribution of Section 2.3.

Definition A.1 (Equilibrium)**.** An *equilibrium* is a tuple

$$\mathcal{E} = \big(\sigma_H, \sigma_L,\ \mu,\ P,\ \bar{\mu},\ \zeta\big),$$

together with the bids of the prepared bidders. Its components are as follows.

- $\sigma_j = \big(\sigma_j(-1), \sigma_j(0), \sigma_j(1)\big)$ is a probability distribution on $\{-1, 0, 1\}$, the order of the informed trader in class $j \in \{H, L\}$. Mixed and wrong-signed orders are allowed.
- $\mu : \{-1, 0, 1\} \to [0, 1]$ is the market makers' posterior $\mu(x) = \Pr(j = H \mid X = x)$.
- $P : \{-1, 0, 1\} \to \mathbb{R}$ is the price after each order.
- $\bar{\mu}$ assigns to each price $\bar{P}$ in the image of P the rival's belief $\Pr(j = H \mid P(X) = \bar{P})$.
- $\zeta(\bar{P}) \in [0, 1]$ is the probability that a rival with the high cost prepares after the price $\bar{P}$. A rival with the zero cost prepares after every price.

For each order x, write $f_j(x) = (1 - \lambda)\eta(x) + \lambda\sigma_j(x)$ for its probability in class j, and $e(x) = \rho + (1 - \rho)\,\zeta(P(x))$ for entry after it. The tuple satisfies six conditions.

(E1) *Bayes consistency.* For every order x, $\mu(x) = \pi f_H(x)/[\pi f_H(x) + (1 - \pi) f_L(x)]$.

(E2) *Beliefs of the rival.* For every price $\bar{P}$ in the image of P, $\bar{\mu}(\bar{P})$ is the posterior of class H given the event $\{x : P(x) = \bar{P}\}$, computed from f_H and f_L.

(E3) *Entry optimality.* For every price $\bar{P}$ in the image, $\zeta(\bar{P}) = 1$ if $B(\bar{\mu}(\bar{P})) > c_H$ and $\zeta(\bar{P}) = 0$ if $B(\bar{\mu}(\bar{P})) < c_H$. At $B(\bar{\mu}(\bar{P})) = c_H$, the rival's tie rule applies.

(E4) *Competitive prices.* For every order x, $P(x) = \mathbb{E}[V_T \mid X = x]$.

(E5) *Optimality of the informed trader.* For each class j, $\sigma_j(q) > 0$ only if q maximizes

$$U_j(q) = q\left[t_0 + e(q)(t_j - t_0) - P(q)\right] - k|q|$$

over $q \in \{-1, 0, 1\}$, with $P(\cdot)$ and $e(\cdot)$ held at their values in $\mathcal{E}$.

(E6) *Truthful bids.* Each prepared bidder whose value meets the reserve bids its value.

The gross profit of a zero-cost rival is an expected auction rent, which is never negative. Preparation is therefore optimal for it after every price, and the definition fixes this choice. The probability $f_j(x)$ is at least $(1 - \lambda)\eta(x) > 0$ under every strategy. Every order therefore occurs with positive probability in both classes. Bayes' rule defines μ after every order, and no off-path belief enters the definition.

The two indifferences in (E3) and (E5) follow the default tie rules of Section 2.4: an indifferent informed trader may mix over its best orders, and an indifferent high-cost rival prepares. Appendix A.4 states the alternatives, under which an indifferent informed trader abstains or an indifferent high-cost rival may prepare with any probability.

A.1 Acquisition payoffs

Truthful bids. Fix a bidder with value θ, and let $Y \geq 0$ be the highest competing admissible bid, with $Y = 0$ when there is none. Any admissible bid that wins pays $\max\{p, Y\}$, so a bid yields either $\theta - \max\{p, Y\}$ or zero. Suppose $\theta \geq p$, and consider the bid θ. If $\theta > \max\{p, Y\}$, it wins and earns $\theta - \max\{p, Y\} > 0$. If $\theta < \max\{p, Y\}$, then $Y > \theta$, and the bid loses and earns zero. If $\theta = \max\{p, Y\}$, a win pays θ, so the bid earns zero.

The bid θ therefore earns $(\theta - \max\{p, Y\})_+$, the larger of the two possible payoffs, at every Y. Suppose $\theta < p$. Every bid that wins pays at least $p > \theta$, so no bid earns more than zero, the payoff of no bid. Truthful bids, and no bid below the reserve, are therefore weakly dominant at every competing bid. Hence condition (E6) of Definition A.1 holds in every tuple. No payoff below rests on a belief.

Proof of Proposition 1. Without entry the target receives $p\mathbf{1}\{R \geq p\}$. A prepared rival with $\theta \geq p$ wins at $\theta > R$ and pays $\max\{p, R\}$. A loss makes the incumbent pay θ. The target receives $\max\{p, \min(R, \theta)\}$, and the rival keeps $(\theta - \max\{p, R\})_+$. At $\theta < p$ the rival does not bid. The target again receives $p\mathbf{1}\{R \geq p\}$, and the rival keeps nothing.

The example. Let R be uniform on $[0, r]$ with $0 < p < \ell < r < h$, as in (7). Then $t_0 = \Pr(R \geq p)\, p = p(1 - p/r)$. Since $R \leq r < h$, the high-class rival always wins and pays $\max\{p, R\}$. The low-class rival wins and pays $\max\{p, R\}$ when $R < \ell$; it loses when $R \geq \ell$.

Integration over R gives

$$\begin{aligned} t_H &= \frac{1}{r}\Big[\int_0^p p\,dy + \int_p^r y\,dy\Big] = \frac{1}{r}\Big[p^2 + \frac{r^2-p^2}{2}\Big] = \frac{r}{2} + \frac{p^2}{2r}, \\ t_L &= \frac{1}{r}\Big[\int_0^p p\,dy + \int_p^\ell y\,dy + \int_\ell^r \ell\,dy\Big] = \frac{1}{r}\Big[p^2 + \frac{\ell^2-p^2}{2} + \ell(r-\ell)\Big] = \ell - \frac{\ell^2-p^2}{2r}, \\ g_H &= \frac{1}{r}\int_0^r \big(h - \max\{p,y\}\big)\,dy = h - t_H = h - \frac{r}{2} - \frac{p^2}{2r}, \\ g_L &= \frac{1}{r}\Big[\int_0^p (\ell-p)\,dy + \int_p^\ell (\ell-y)\,dy\Big] = \frac{1}{r}\Big[p(\ell-p) + \frac{(\ell-p)^2}{2}\Big] = \frac{\ell^2-p^2}{2r}. \end{aligned} \tag{A.1}$$

The spread is

$$\Delta_T(r) = t_H - t_L = \frac{r}{2} + \frac{p^2}{2r} - \ell + \frac{\ell^2-p^2}{2r} = \frac{r^2-2\ell r+\ell^2}{2r} = \frac{(r-\ell)^2}{2r}.$$

The sums are $t_H + g_H = h$ and $t_L + g_L = \ell$, as the split requires. These are the expressions in (8).

Differentiation in r gives

$$\Delta_T'(r) = \frac{2(r-\ell)\cdot 2r - 2(r-\ell)^2}{4r^2} = \frac{(r-\ell)(r+\ell)}{2r^2} = \frac{1}{2} - \frac{\ell^2}{2r^2},$$
$$g_H'(r) = -\frac{1}{2} + \frac{p^2}{2r^2}, \qquad g_L'(r) = -\frac{\ell^2-p^2}{2r^2}.$$

The signs in (9) follow from $r > \ell$, $r > p$, and $\ell > p$. For every $\mu \in [0,1]$, $\partial B_r(\mu)/\partial r = (1-\mu)g_L'(r) + \mu g_H'(r) < 0$.

General classes and a continuous F. Write the payments as integrals of indicators. For $\theta \geq p$,

$$\max\{p, \min(R,\theta)\} = p + \int_p^\theta \mathbf{1}\{R > y\}\,dy, \qquad (\theta - \max\{p,R\})_+ = \int_p^\theta \mathbf{1}\{R < y\}\,dy. \tag{A.2}$$

The two indicators sum to one except at the single point $y = R$, which gives $t(\theta) + g(\theta) = \theta$.

Every integrand in (A.2) is nonnegative and jointly measurable in (R, y). Tonelli's theorem therefore moves the expectation over R inside each integral. Since F is continuous, $\Pr(R > y) = 1 - F(y)$. This gives $t(\theta) = p + \int_p^\theta [1 - F(y)]\,dy$ and $g(\theta) = \int_p^\theta F(y)\,dy$ for $\theta \geq p$. For $\theta < p$, $t(\theta) = t_0 = p[1 - F(p)]$ and $g(\theta) = 0$.

At $\theta = p$, t jumps by $pF(p)$. When the incumbent fails the reserve, an admissible rival buys at p, and an inadmissible one leaves the target unsold. Hence, for every θ,

$$t(\theta) = t_0 + pF(p)\mathbf{1}\{\theta \geq p\} + \int_p^\infty \mathbf{1}\{y \leq \theta\}[1 - F(y)]\,dy, \qquad g(\theta) = \int_p^\infty \mathbf{1}\{y \leq \theta\}F(y)\,dy.$$

Take the class expectation and apply Tonelli's theorem again, with $\mathbb{E}_j \mathbf{1}\{\theta \geq p\} = 1 - G_j(p^-)$ and $\Pr_j(\theta \geq y) = 1 - G_j(y^-)$. The functions $G_j(y^-)$ and $G_j(y)$ differ at countably many y, a Lebesgue-null set, so the integral is the same with $G_j(y)$ in place of $G_j(y^-)$. This gives

$$t_j = t_0 + pF(p)[1 - G_j(p^-)] + \int_p^\infty [1 - G_j(y)][1 - F(y)]\, dy$$

and g_j in (5). The difference of the two classes gives Δ_T in (5). Each integral is finite, since $0 \leq \int_p^\infty [1 - G_j(y)]\, dy \leq \mathbb{E}_j \theta < \infty$ and $0 \leq F \leq 1$.

Since $\mathbb{E}_j[\max\{\theta, p\}] = p + \int_p^\infty [1 - G_j(y)]\, dy$, the sum of the two lines of (5) gives the split in its general form,

$$\Delta_T(F) + g_H(F) - g_L(F) = pF(p)D(p^-) + \mathbb{E}_H[\max\{\theta, p\}] - \mathbb{E}_L[\max\{\theta, p\}].$$

The reserve term vanishes under equal admissibility, $G_H(p^-) = G_L(p^-)$. It also vanishes when the incumbent always meets the reserve, $F(p) = 0$.

Part (a). In (5), the weight $1 - G_j$ on F is nonnegative. The pointwise decrease from F to $\tilde{F}$ therefore lowers g_j by $\int_p^\infty [1 - G_j(y)][F(y) - \tilde{F}(y)]\, dy \geq 0$. The decrease is strict exactly when this integral is positive. Gross profit at a fixed belief μ is $(1 - \mu)g_L + \mu g_H$, an average with fixed weights, so it falls as well.

Part (b). The difference of (5) at the two distributions is

$$\Delta_T(\tilde{F}) - \Delta_T(F) = \int_p^\infty [F(y) - \tilde{F}(y)]\, D(y)\, dy - p[F(p) - \tilde{F}(p)]\, D(p^-).$$

Under the class ranking (1), $D(p^-) = 0$, so the second term is zero. Also $D(y) \geq 0$ and $F(y) - \tilde{F}(y) \geq 0$ for every y, so the integrand is nonnegative. The spread therefore rises, strictly exactly when the integral is positive.

Equal admissibility also makes the second term zero. Together with $D(y) \geq 0$ for $y > p$, it gives the same conclusion. □

Proposition 1(c) (the weakest restriction). *Let G_H and G_L have finite means. The spread weakly increases under every first-order stochastic strengthening of every continuous F if and only if $D(y) \geq 0$ for every $y > p$ and $D(p^-) \leq 0$. When neither class has an atom at p, this condition is exactly first-order dominance above the reserve together with equal admissibility, $G_H(p^-) = G_L(p^-)$.*

Proof of Proposition 1(c). Fix the incumbent's value R. Suppose $R \geq p$. The target receives $\min\{R, \max\{\theta, p\}\}$ for every value of the rival. An admissible rival and the incumbent both bid

at least p, and an inadmissible rival leaves the incumbent to pay p. Suppose $R < p$. The target receives $p\mathbf{1}\{\theta \geq p\}$.

For $R \geq p$, $\mathbb{E}_j[\min\{R, \max\{\theta, p\}\}] = p + \int_p^R [1 - G_j(y)]\, dy$. The expected difference in proceeds between the classes at incumbent value R is therefore

$$\Gamma(R) = \begin{cases} pD(p^-), & R < p, \\ \int_p^R D(y)\, dy, & R \geq p, \end{cases}$$

and $\Delta_T(F) = \mathbb{E}_F[\Gamma(R)]$.

The function Γ is bounded, since $\int_p^\infty |D(y)|\, dy \leq \mathbb{E}_H\theta + \mathbb{E}_L\theta < \infty$. It is constant below p, continuous on $[p, \infty)$, and has derivative D almost everywhere above p. It moves from $pD(p^-)$ to $\Gamma(p) = 0$ at p. Thus Γ is nondecreasing if and only if $D \geq 0$ almost everywhere above p and $D(p^-) \leq 0$.

If Γ is nondecreasing, every first-order stochastic strengthening raises $\mathbb{E}_F[\Gamma(R)]$. If it is not, continuity on each side of p gives values $0 < R_1 < R_2$, both different from p, with $\Gamma(R_1) > \Gamma(R_2)$. Let F be uniform on $[R_1 - \delta, R_1 + \delta]$, and let $\tilde{F}$ be the same distribution shifted to R_2. Then $\tilde{F} \leq F$, and as $\delta \to 0$, $\Delta_T(\tilde{F}) - \Delta_T(F) \to \Gamma(R_2) - \Gamma(R_1) < 0$.

Suppose the support must start at zero. Mix the same small uniform component on $[0, \bar{r}]$ into both distributions. Both mixtures are continuous, and the mixture scales the difference by a positive factor and keeps its sign.

Distribution functions are right-continuous. Thus $D \geq 0$ almost everywhere above p is the same as $D(y) \geq 0$ for every $y > p$, and it implies $D(p) \geq 0$. When neither class has an atom at p, $D(p^-) = D(p) \geq 0$. The condition then holds exactly when $D(p^-) = 0$, which with the ranking above the reserve is equal admissibility. □

Scope of the class ranking. The proofs in this subsection use the class ranking (1) only through $D(y) \geq 0$ for $y > p$ and $D(p^-) = 0$. These two facts give $g_H \geq g_L$ and $\Delta_T \geq 0$ in (5), the comparison in part (b), and the split identity of Section 3.1. The same results therefore hold under the ranking above the reserve together with equal admissibility.

For the point masses with $p < \ell$, $D(y) = \mathbf{1}\{\ell \leq y < h\}$ and $D(p^-) = 0$. Also $\Gamma(R) = (R - \ell)_+$ for $R < h$. Equation (5) then gives $\Delta_T = \int_\ell^h [1 - F(y)]\, dy$, which equals $\int_\ell^{\bar{r}} [1 - F(y)]\, dy$ at $\bar{r} < h$. It also gives $g_j = \int_p^\theta F(y)\, dy$ at $\theta = h$ and $\theta = \ell$.

For the uniform distribution on $[0, r]$, these integrals reproduce (A.1). The kernel Γ belongs to the cash second-price rule with reserve. Section 7.2 changes the rule and the kernel with it.

A.2 Beliefs and trading

Beliefs, prices and trading. The belief lemma holds for every order profile, in or out of equilibrium; the price and trading lemmas hold for every tuple that meets their stated conditions.

Conventions for Appendices A.2 to A.9. The conditions (E1) to (E6) are those of Definition A.1.

- A *profile* is a pair (σ_H, σ_L) of order distributions, pure or mixed, with wrong-signed orders allowed. A *candidate* is a tuple $\mathcal{E}$ that satisfies (E1) to (E4) and (E6) for some profile. An equilibrium is a candidate that also satisfies (E5).
- A deviation by the informed trader holds the candidate's price function P and preparation rule ζ fixed and changes only the order that reaches them; a comparison across incumbents solves the equilibrium in each economy to compare outcomes across economies.
- A uniqueness proof derives its bounds for every candidate. It does not use the proposed tuple.
- A constructed tuple satisfies each equilibrium condition by direct substitution, and its proof cites no lemma whose premises include the condition at issue.
- Write $w_j = t_j - t_0$ for the gain in expected proceeds that entry brings in class j, and $w(\mu) = (1-\mu)w_L + \mu w_H = t_L - t_0 + \Delta_T \mu$ for its average at the belief μ. Proposition 1 and Section 3.1 give $w_H \geq w_L \geq 0$.

Orders and posteriors. Write δ_q for the point mass at the order q, and $b_0 = (1-\lambda)(1-2\beta)$ for the probability of a liquidity abstention. Under any order distributions σ_H, σ_L, Bayes' rule in odds form reads

$$\frac{\mu(x)}{1-\mu(x)} = \frac{\pi}{1-\pi} \cdot \frac{f_H(x)}{f_L(x)}. \tag{A.3}$$

Lemma A.1 (Order probabilities and beliefs)**.** *Fix any order distributions σ_H and σ_L.*

(a) Full support. $f_j(x) \geq (1-\lambda)\eta(x) > 0$ *for every order x and class j.*

(b) Nonzero orders. *For $x \in \{-1, 1\}$, $b \leq f_j(x) \leq b + \lambda$. Hence $m(\pi) \leq \mu(x) \leq M(\pi)$.*

(c) The zero order. $b_0 \leq f_j(0) \leq b_0 + \lambda$. *If $\beta \leq 1/3$, then $b_0 \geq b$, hence $m(\pi) \leq \mu(0) \leq M(\pi)$.*

(d) Full orders. *If $\sigma_H = \delta_1$ and $\sigma_L = \delta_{-1}$, the posteriors after the orders $(-1, 0, 1)$ are $\big(m(\pi), \pi, M(\pi)\big)$.*

Proof of Lemma A.1. Fix any profile (σ_H, σ_L). By (A.3), the posterior $\mu(x)$ increases strictly in the likelihood ratio $f_H(x)/f_L(x)$. The odds of $M(\pi)$ are $\pi(1+u)/(1-\pi)$, and the odds of $m(\pi)$ are $\pi/[(1-\pi)(1+u)]$.

Part (a). The liquidity trader fills the slot with probability $1-\lambda > 0$. It places the order x with probability $\eta(x)$, and $\eta(x) > 0$ for every x because $0 < \beta < 1/2$. The informed term $\lambda\sigma_j(x)$ is nonnegative. Hence $f_j(x) \geq (1-\lambda)\eta(x) > 0$ for every order and class. Every order has positive probability in both classes, so (E1) defines $\mu(x)$ at every order, and no off-path belief enters any condition. Also $0 < \mu(x) < 1$, since $0 < \pi < 1$.

Part (b). For $x = \pm 1$, $(1-\lambda)\eta(x) = b$ and $0 \leq \lambda\sigma_j(x) \leq \lambda$. Hence $b \leq f_j(x) \leq b + \lambda$, and

$$\frac{1}{1+u} = \frac{b}{b+\lambda} \leq \frac{f_H(x)}{f_L(x)} \leq \frac{b+\lambda}{b} = 1+u.$$

By (A.3), $m(\pi) \leq \mu(x) \leq M(\pi)$. This step uses only $0 < \beta < 1/2$.

Part (c). For $x = 0$, $(1-\lambda)\eta(0) = b_0$, so $b_0 \leq f_j(0) \leq b_0 + \lambda$, and the likelihood ratio lies in $[(1+\lambda/b_0)^{-1}, 1+\lambda/b_0]$. The inequality $b_0 \geq b$ is $1 - 2\beta \geq \beta$, that is $\beta \leq 1/3$. Then $\lambda/b_0 \leq \lambda/b = u$, and the argument of part (b) gives $m(\pi) \leq \mu(0) \leq M(\pi)$. If $\beta > 1/3$, then $b_0 < b$. The profile $\sigma_H = \delta_0$ with $\sigma_L(0) = 0$ gives $f_H(0) = b_0 + \lambda$ and $f_L(0) = b_0$. The ratio is $1 + \lambda/b_0 > 1 + u$, so $\mu(0) > M(\pi)$, and the bound fails.

Part (d). The profile $\sigma_H = \delta_1$, $\sigma_L = \delta_{-1}$ gives $f_H = (b,\, b_0,\, b+\lambda)$ and $f_L = (b+\lambda,\, b_0,\, b)$ on the orders $(-1, 0, 1)$. The likelihood ratios are $(1+u)^{-1}$, 1 and $1+u$, so $(\mu(-1), \mu(0), \mu(1)) = (m(\pi), \pi, M(\pi))$.

Beliefs at a price. Fix a price $\bar{P}$ in the image of P, and let $X(\bar{P}) = \{x : P(x) = \bar{P}\}$. By (E2), the rival's belief $\bar{\mu}(\bar{P})$ has the odds

$$\frac{\bar{\mu}(\bar{P})}{1-\bar{\mu}(\bar{P})} = \frac{\pi}{1-\pi} \cdot \frac{\sum_{x \in X(\bar{P})} f_H(x)}{\sum_{x \in X(\bar{P})} f_L(x)}.$$

A ratio of two sums of positive terms lies between the smallest and the largest ratio of its terms. Hence $\bar{\mu}(\bar{P})$ lies between the smallest and the largest $\mu(x)$ over $X(\bar{P})$. Every bound of parts (b) and (c) therefore holds for $\bar{\mu}(\bar{P})$.

Only part (c) uses $\beta \leq 1/3$. Parts (a), (b) and (d) hold for every $\beta < 1/2$. □

Corollary A.1 (High-cost preparation under every profile)**.** *Let $0 < \beta \leq 1/3$ and $g_H \geq g_L$. If $B(M(\pi)) < c_H$, then no profile that satisfies (E1)–(E3), in or out of equilibrium, has high-cost preparation after any order.*

Proof of Corollary A.1. Let $\beta \leq 1/3$, $g_H \geq g_L$ and $B(M(\pi)) < c_H$. Fix any profile and any tuple that satisfies (E1) to (E3). By Lemma A.1(b)–(c), $\mu(x) \leq M(\pi)$ at every order, and by the step

on beliefs at a price, $\bar{\mu}(\bar{P}) \le M(\pi)$ at every price. Since $g_H \ge g_L$, $B(\mu) = g_L + \mu(g_H - g_L)$ is nondecreasing in μ. Hence $B(\bar{\mu}(\bar{P})) \le B(M(\pi)) < c_H$ at every price, and (E3) gives $\zeta(\bar{P}) = 0$ with no tie. The conclusion holds under both tie rules for the rival. The bound at the zero order is the only use of $\beta \le 1/3$ in this proof. □

Lemma A.2 (Entry and prices)**.** *Let $\Delta_T > 0$ and $g_H > g_L$. Fix a tuple that satisfies (E1)–(E4) and (E6).*

(a) Floor. *$e(x) \ge \rho$ for every order x.*

(b) Price. *$P(x) = t_0 + e(x)\big[t_L - t_0 + \Delta_T\, \mu(x)\big]$ for every order x.*

(c) Sufficiency. *If $P(x) = P(x')$, then $\mu(x) = \mu(x')$. The rival's belief at the price $P(x)$ is $\mu(x)$.*

(d) Entry. *$e(x) = \rho$ if $\mu(x) < \tau$, and $e(x) = 1$ if $\mu(x) \ge \tau$.*

(e) Monotonicity. *If $\mu(x) > \mu(x')$, then $e(x) \ge e(x')$ and $P(x) - P(x') \ge \rho\Delta_T\,[\mu(x) - \mu(x')] > 0$.*

When an indifferent high-cost rival may prepare with any probability, part (d) at $\mu(x) = \tau$ allows any $e(x) \in [\rho, 1]$; the other parts hold under both tie rules for the rival.

Proof of Lemma A.2. Fix a candidate, and let $\Delta_T > 0$ and $g_H > g_L$. The proof has a preliminary step and five parts.

Step 0: the terminal payoff given the class and the order. The rival's choice depends only on the price $P(x)$ and its cost C. The cost is independent of the class, the values, the incumbent's value and the slot draws. Given the class j and the order x, the rival therefore prepares with probability $e(x) = \rho + (1 - \rho)\zeta(P(x))$, and this event is independent of (θ, R). The order depends on the class and the slot draws only, so it carries no information about (θ, R) given j. Bids are truthful by (E6). With entry, expected proceeds in class j are t_j; without entry, they are t_0. Hence

$$\mathbb{E}[V_T \mid j, X = x] = t_0 + e(x)\, w_j. \tag{A.4}$$

Part (a): the floor. The zero-cost rival prepares after every price (Definition A.1), and it has probability ρ. Hence $e(x) \ge \rho$ at every order.

Part (b): the price. Average (A.4) over the class with the weights $\mu(x)$ and $1 - \mu(x)$ of (E1). Condition (E4) gives $P(x) = t_0 + e(x)\, w(\mu(x))$, which is (4). This step uses only (E1), (E4), the truthful bids (E6) of Step 0, and the independence of the cost.

Part (c): sufficiency. Let $P(x) = P(x')$. Entry depends on the order only through the price, so $e(x) = e(x') =: \bar{e}$, and $\bar{e} \ge \rho > 0$ by part (a). Part (b) gives $\bar{e}\,\Delta_T\,[\mu(x) - \mu(x')] = 0$, so $\mu(x) = \mu(x')$. By the step on beliefs at a price in the proof of Lemma A.1, $\bar{\mu}(P(x))$ lies between

the smallest and the largest posterior of the orders with the price $P(x)$. These posteriors are equal, so $\bar{\mu}(P(x)) = \mu(x)$. The price therefore reveals the posterior.

Part (d): entry. Since $g_H > g_L$, $B(\mu) > c_H$ exactly when $\mu > \tau$. By part (c), the rival's gross profit at $P(x)$ is $B(\mu(x))$. Condition (E3) gives $e(x) = \rho$ if $\mu(x) < \tau$ and $e(x) = 1$ if $\mu(x) > \tau$. At $\mu(x) = \tau$, $e(x) = 1$ when an indifferent high-cost rival prepares, and $e(x)$ can take any value in $[\rho, 1]$ when an indifferent high-cost rival may prepare with any probability.

Part (e): monotonicity. Let $\mu(x) > \mu(x')$. If $\mu(x') < \tau$, then $e(x') = \rho \leq e(x)$. If $\mu(x') \geq \tau$, then $\mu(x) > \tau$ and $e(x) = 1 \geq e(x')$. Part (b) gives

$$P(x) - P(x') = e(x)\,\Delta_T\,[\mu(x) - \mu(x')] + [e(x) - e(x')]\,w(\mu(x')). \tag{A.5}$$

The second term is nonnegative, because $w(\mu) \geq w_L \geq 0$. The first term is at least $\rho\Delta_T[\mu(x) - \mu(x')] > 0$. Prices therefore rise strictly with the posterior. □

Constructed tuples. Let $\Delta_T > 0$. Take a profile, its Bayes posteriors $\mu(x)$, and entry $e(x) \in [\rho, 1]$ such that $e(x) \geq e(x')$ whenever $\mu(x) > \mu(x')$. Set $P(x) = t_0 + e(x)[w_L + \Delta_T\mu(x)]$, where $w_L = t_L - t_0 \geq 0$. Then prices rise strictly with the posterior, and

$$P(x) = P(x') \quad \Longleftrightarrow \quad \mu(x) = \mu(x') \text{ and } e(x) = e(x').$$

The preparation rule $\zeta(P(x)) = [e(x) - \rho]/(1 - \rho)$ is a function of the price, and the belief at $P(x)$ is $\mu(x)$. If also $g_H > g_L$, threshold-consistent entry is a function of the posterior when an indifferent high-cost rival prepares. When an indifferent high-cost rival may prepare with any probability, orders at τ with different entry have different prices, each revealing τ.

Proof. Let $\mu(x) > \mu(x')$. Then $e(x) \geq e(x')$, and (A.5) gives $P(x) - P(x') \geq e(x)\Delta_T[\mu(x) - \mu(x')] \geq \rho\Delta_T[\mu(x) - \mu(x')] > 0$, because $w(\mu) \geq w_L \geq 0$. Let $\mu(x) = \mu(x')$. Then $P(x) - P(x') = [e(x) - e(x')]\,w(\mu(x))$, and $w(\mu(x)) = w_L + \Delta_T\mu(x) > 0$, because $\mu(x) > 0$ by Lemma A.1(a). Equal prices therefore force equal posteriors and equal entry, and the price formula gives the converse. So ζ is a function of the price, and $e(x) \in [\rho, 1]$ puts its values in $[0, 1]$. At a price $\bar{P}$, all orders with the price $\bar{P}$ share one posterior, so by (A.3) they share one likelihood ratio $f_H(x)/f_L(x)$. The ratio of sums in the step on beliefs at a price, in the proof of Lemma A.1, equals that likelihood ratio, and (E2) gives the common posterior. Under the first tie rule for the rival, threshold-consistent entry is ρ below τ and one at and above τ, a function of the posterior. Under the second, the identity at equal posteriors gives distinct prices to distinct entries at τ, and every order with one of these prices has the posterior τ. □

Lemma A.3 (Informed trading)**.** *Let $\Delta_T \geq 0$. Fix a tuple that satisfies (E1), (E4) and (E6), with entry $e(x) \in [\rho, 1]$ after every order. Conditional on its arrival, the informed trader in each class earns*

$$U_H(1) = e(1)\Delta_T[1-\mu(1)] - k, \qquad U_H(0) = 0, \qquad U_H(-1) = -e(-1)\Delta_T[1-\mu(-1)] - k,$$
$$U_L(-1) = e(-1)\Delta_T\,\mu(-1) - k, \qquad U_L(0) = 0, \qquad U_L(1) = -e(1)\Delta_T\,\mu(1) - k.$$

Write $K_H = e(1)\Delta_T[1-\mu(1)]$ and $K_L = e(-1)\Delta_T\,\mu(-1)$ for the gross returns of the two correct orders.

(a) A wrong-signed order earns at most $-k$, so abstention strictly beats it.

(b) Class H buys if $K_H > k$ and abstains if $K_H < k$. At $K_H = k$, the purchase and abstention are its two best orders. Class L sells if $K_L > k$ and abstains if $K_L < k$. At $K_L = k$, the sale and abstention are its two best orders.

(c) Under every candidate profile, $\rho\Delta_T[1-M(\pi)] \leq K_H \leq \Delta_T$ and $\rho\Delta_T\,m(\pi) \leq K_L \leq \Delta_T$. Hence $\rho\,\kappa(\pi)\Delta_T \leq K_j \leq \Delta_T$, where $\kappa(\pi) = \min\{m(\pi), 1-M(\pi)\}$.

(d) K_H depends on the candidate only through $\mu(1)$ and $e(1)$, and K_L only through $\mu(-1)$ and $e(-1)$.

Proof of Lemma A.3. Fix a tuple that satisfies (E1), (E4) and (E6), with $e(x) \in [\rho, 1]$ after every order, and let $\Delta_T \geq 0$. A deviation to the order q reaches the tuple's price $P(q)$ and entry $e(q)$. By (A.4) and part (b) of the proof of Lemma A.2,

$$\mathbb{E}[V_T \mid H, X = q] - P(q) = e(q)\,[w_H - w(\mu(q))] = e(q)\,\Delta_T\,[1-\mu(q)],$$
$$P(q) - \mathbb{E}[V_T \mid L, X = q] = e(q)\,[w(\mu(q)) - w_L] = e(q)\,\Delta_T\,\mu(q).$$

The proceeds t_0 and the term $e(q)w_L$ cancel. Multiply by q and subtract $k|q|$. This gives the six payoffs of the lemma, with $K_H = e(1)\Delta_T[1-\mu(1)]$ and $K_L = e(-1)\Delta_T\mu(-1)$.

Part (a). Since $e \geq 0$, $\Delta_T \geq 0$ and $0 \leq \mu \leq 1$, the payoffs $U_H(-1) = -e(-1)\Delta_T[1-\mu(-1)] - k$ and $U_L(1) = -e(1)\Delta_T\mu(1) - k$ are at most $-k < 0 = U_j(0)$. No class puts weight on a wrong-signed order under (E5).

Part (b). By part (a), class H compares the purchase with abstention, and $U_H(1) - U_H(0) = K_H - k$. Class H buys with probability one if $K_H > k$ and abstains if $K_H < k$. If $K_H = k$, the purchase and abstention are its two maximizers: an indifferent informed trader may mix over them or abstains. Class L compares the sale with abstention, and $U_L(-1) - U_L(0) = K_L - k$. Class L sells with probability one if $K_L > k$ and abstains if $K_L < k$. If $K_L = k$, the sale and abstention are its two maximizers: an indifferent informed trader may mix over them or abstains.

Part (c). By Lemma A.1(b), $\mu(1) \le M(\pi)$ and $\mu(-1) \ge m(\pi)$ under every profile, for every $\beta < 1/2$. Also $\rho \le e \le 1$ and $0 \le \mu \le 1$. Hence

$$\rho\,\kappa(\pi)\,\Delta_T \le \rho\,[1 - M(\pi)]\,\Delta_T \le K_H \le \Delta_T, \qquad \rho\,\kappa(\pi)\,\Delta_T \le \rho\,m(\pi)\,\Delta_T \le K_L \le \Delta_T. \tag{A.6}$$

The upper bounds use only $e \le 1$ and $0 \le \mu \le 1$.

Part (d). The return K_H depends on the tuple only through $(\mu(1), e(1))$, and K_L only through $(\mu(-1), e(-1))$. Entry after the zero order enters no payoff. The ex ante payoff of the informed trader is $\lambda\,U_j(q)$, and the factor $\lambda > 0$ changes no comparison. □

Lemma A.4 (The auction through the spread and the threshold)**.** *Fix the model's independence and information structure, a prior $\pi \in (0,1)$, the two-point preparation cost with $\rho \in (0,1)$ and $c_H > 0$, a trading cost $k > 0$, and one pair of tie rules: an indifferent informed trader may mix over its best orders or abstains, and an indifferent high-cost rival prepares or may prepare with any probability. At fixed presence, fix $0 < \lambda < 1$; with research, presence is $\hat{\lambda}$. Fix $0 < \beta < 1/2$. Consider two cash second-price auctions with finite-mean classes satisfying the class ranking, continuous incumbent distributions, $\Delta_T > 0$ and $\gamma = g_H - g_L > 0$, and the same spread and preparation threshold $\tau = (c_H - g_L)/\gamma$. Their equilibrium sets projected onto $(\sigma_H, \sigma_L, \mu, e)$ coincide.*

The common conditions are Bayes' rule, the threshold entry rule under the chosen tie rule, and optimal orders at

$$\begin{aligned} U_H(1) &= e(1)\Delta_T[1-\mu(1)] - k, \quad U_H(-1) = -e(-1)\Delta_T[1-\mu(-1)] - k, \\ U_L(-1) &= e(-1)\Delta_T\mu(-1) - k, \qquad U_L(1) = -e(1)\Delta_T\mu(1) - k, \end{aligned} \qquad U_j(0) = 0.$$

Each auction gives prices and price beliefs by

$$P(x) = t_0 + e(x)[w_L + \Delta_T\mu(x)], \qquad \zeta(P(x)) = \frac{e(x) - \rho}{1 - \rho}, \qquad \bar{\mu}(P(x)) = \mu(x).$$

With research, fix also the quadratic cost coefficient $a > 0$ and cap $0 < \bar{\lambda} < 1$, and require optimal orders after a hypothetical success even at zero research. The equilibrium sets projected onto $(\hat{\lambda}, \sigma_H, \sigma_L, \mu, e)$ coincide. The common conditions also include

$$\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\}, \qquad \Psi = \pi \max_q U_H(q) + (1-\pi)\max_q U_L(q),$$

with Bayes' rule evaluated at $\hat{\lambda}$. These conclusions hold under all four tie rules.

Proof. For either auction, Lemma A.2 makes the price posterior equal to the order posterior. Since $B(\mu) - c_H = \gamma(\mu - \tau)$, preparation optimality is the same threshold rule in both auctions, including

the selected action at equality. Subtracting the competitive price from the class payoff gives the four nonzero-order payoffs in the statement by Lemma A.3. The price levels t_0 and w_L cancel. Thus every equilibrium gives a solution of the same reduced conditions.

Conversely, take a solution of those conditions. Threshold-consistent entry is nondecreasing across distinct posteriors. The constructed-tuples paragraph supplies the displayed prices, a preparation rule that is a function of price, and the price beliefs. In particular, orders with a common price have a common posterior and common entry; orders at the preparation threshold with different entry have different prices. Bayes' rule at a pooled price follows by summing order probabilities with the same likelihood ratio. Preparation is optimal because $\gamma > 0$, competitive pricing follows from the conditional terminal payoff, and optimal orders are exactly the reduced payoff comparisons. Truthful bidding completes the tuple. This construction works in either auction.

With research, at the candidate prices and preparation rule a deviation to intensity $\tilde{\lambda}$ and optimal success-contingent orders earns

$$\tilde{\lambda}\big[\pi \max_q U_H(q) + (1-\pi)\max_q U_L(q)\big] - \frac{a}{2}\tilde{\lambda}^2.$$

Its strict concavity on $[0, \bar{\lambda}]$ gives $\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\}$. At zero intensity the stipulated hypothetical-success optimality gives the same maximizers. Since the payoffs, coefficient and cap coincide, the research condition and the price construction coincide as well. □

Lemma A.5 (The two numbers along a path)**.** *Fix ranked finite-mean classes, the reserve and the two-point preparation cost. A path is a family F_r of continuous incumbent distributions indexed by strength on an interval, with $F_{r'} \le F_r$ when $r' > r$. Write $\Delta_T(r) = \Delta_T(F_r)$, $\gamma(r) = g_H(F_r) - g_L(F_r)$, and, where $\gamma(r) > 0$, $\tau(r) = [c_H - g_L(F_r)]/\gamma(r)$. The class mean gap is fixed:*

$$\bar{\Gamma} = \mathbb{E}_H\theta - \mathbb{E}_L\theta = \Delta_T(r) + \gamma(r).$$

Each g_j and γ is nonincreasing, and Δ_T is nondecreasing. For $r' > r$,

$$\Delta_T(r') - \Delta_T(r) = \int_p^\infty [F_r(y) - F_{r'}(y)]D(y)\,dy,$$
$$g_j(F_r) - g_j(F_{r'}) = \int_p^\infty [F_r(y) - F_{r'}(y)][1 - G_j(y)]\,dy.$$

Each change is strict exactly when its integral is positive. Where $\gamma > 0$, $\Delta_T > 0$ and $\tau \ge 0$, both τ and $\Delta_T\tau$ are nondecreasing. Their exact increments are

$$\tau(r') - \tau(r) = \frac{g_L(F_r) - g_L(F_{r'}) + \tau(r)[\Delta_T(r') - \Delta_T(r)]}{\gamma(r')},$$

$$\Delta_T(r')\tau(r') - \Delta_T(r)\tau(r) = \Delta_T(r)[\tau(r') - \tau(r)] + \tau(r')[\Delta_T(r') - \Delta_T(r)].$$

Thus the threshold rises strictly exactly when the numerator in its increment is positive; the product rises strictly exactly when the right side of its increment is positive.

On a differentiable path the return $J = \Delta_T(1 - \tau)$ obeys

$$J' = \frac{\bar{\Gamma}(g_H - c_H)\Delta_T' + \Delta_T \gamma g_H'}{\gamma^2}.$$

Its direction depends on the balance between spread growth and the decline in high-class profit. For point classes $h = 10$, $\ell = 1$, reserve $p = 1/2$, high cost $c_H = 2$, and $F_r = U[r - 1/10, r + 1/10]$ on $[7, 15/2]$,

$$\Delta_T = r - \ell, \qquad g_L = 0, \qquad g_H = h - r, \qquad \tau = \frac{c_H}{h - r}.$$

Its return falls between the endpoints:

$$J(7) = 2,$$
$$J(15/2) = 13/10 < J(7).$$

In the uniform-incumbent example below, J rises on $[r_0, r_2]$ by the uniform case of Proposition 3(c).

Proof. The split identity $t_j(F) + g_j(F) = \mathbb{E}_j\theta$ gives $\Delta_T + \gamma = \bar{\Gamma}$. The auction integrals are

$$g_j(F) = \int_p^\infty F(y)[1 - G_j(y)]\, dy, \qquad \gamma(F) = \int_p^\infty F(y)D(y)\, dy.$$

Subtracting the integrals at two ordered distributions gives the stated profit differences. Subtracting the profit-gap integrals and using the split identity gives the spread difference. Their integrands are nonnegative, so the monotonicity and exact strictness conditions follow.

Because $\gamma(r') = \gamma(r) - [\Delta_T(r') - \Delta_T(r)]$, direct subtraction of the threshold ratios gives its displayed increment. Its numerator is nonnegative when $\tau(r) \geq 0$, and its denominator is positive. Expanding $\Delta_T(r')\tau(r') - \Delta_T(r)\tau(r)$ gives the product increment. Both terms are nonnegative, which proves the product comparison and its exact strictness condition.

On a differentiable path, write $1 - \tau = (g_H - c_H)/\gamma$ and $\gamma' = -\Delta_T'$. Differentiation gives

$$J' = \frac{(g_H - c_H)\Delta_T' + \Delta_T g_H'}{\gamma} + \frac{\Delta_T(g_H - c_H)\Delta_T'}{\gamma^2}.$$

Collecting the Δ_T' terms and using $\gamma + \Delta_T = \bar{\Gamma}$ gives the stated expression. For the translated uniform distributions every incumbent value lies strictly between ℓ and h. Thus $t_L = \ell$, $t_H = \mathbb{E}R = r$, $g_L = 0$ and $g_H = h - r$, which gives all four formulas and the endpoint values of J by substitution. In

the uniform-incumbent case, Proposition 3(c) gives that $\Delta_T(g_H - c_H)$ is nondecreasing. Since $J = \Delta_T(g_H - c_H)/\gamma$, $\gamma = \bar{\Gamma} - \Delta_T$ falls strictly where Δ_T rises, and affordability gives $g_H > c_H$, J rises on $[r_0, r_2]$. □

The liquidity share. The bound $\beta \leq 1/3$ is the inequality $b_0 \geq b$. It extends the posterior bounds of the nonzero orders to the zero order, and through it enters only Corollary A.1 and the zero-order part of the research bounds. The research parameter region and the slot-feasibility parts of identification range over the slots with $\beta \leq 1/3$. Every other result holds for every $0 < \beta < 1/2$; a corollary that fixes the slot holds at that slot.

A price reached by a nonzero order. For every profile and every price function, a price reached by at least one nonzero order has a price posterior in $[m(\pi), M(\pi)]$, for every $0 < \beta < 1/2$, including when the price pools the zero order.

Proof. Let $X(\bar{P}) = \{x : P(x) = \bar{P}\}$, and suppose it contains a nonzero order. Its liquidity mass $(1 - \lambda) \sum_{x \in X(\bar{P})} \eta(x)$ is at least b. Its informed mass in each class lies between zero and λ. Thus the ratio of its class probabilities lies between $(1 + u)^{-1}$ and $1 + u$. Bayes' rule gives the stated interval. □

A.3 Competition creates competition

Proof of Proposition 2. Write F_1 for a strengthening of F_0 and F_2 for a strengthening of F_1. For $i \in \{0, 1, 2\}$, write $B_i = B_{F_i}$, $\Delta_i = \Delta_T(F_i)$, and $t_{0,i}$, $t_{j,i}$, $g_{j,i}$, $w_i(\mu)$ for the objects at F_i. Also write $M = M(\pi)$, $m = m(\pi)$ and $\kappa = \kappa(\pi)$. Under the class ranking (1) every value meets the reserve, so $W_H(F) = \mathbb{E}_H[F(\theta)]$, the probability that a prepared high-class rival outbids the incumbent. Value ties have probability zero because F is continuous.

The proof has seven steps. Steps 1, 3 and 5 state each economy's equilibrium tuple and derive its equilibrium conditions. Bounds on every candidate give uniqueness.

Step 0: implied inequalities. Proposition 1(a) gives $B_1(\pi) \leq B_0(\pi)$, so (H1) gives $B_1(\pi) < c_H < B_1(M)$. Since $M > \pi$, $B_1(M) - B_1(\pi) = (M - \pi)(g_{H,1} - g_{L,1}) > 0$. Hence $g_{H,1} > g_{L,1}$, the threshold τ_1 at F_1 is defined, and $\pi < \tau_1 < M$. By (H2), $\Delta_1 > k/(\rho\kappa) > 0$. Proposition 1(b) gives $\Delta_2 \geq \Delta_1$, so $\rho\kappa\Delta_2 > k$. Proposition 1 also gives $g_{H,2} \geq g_{L,2}$, so $B_2(\mu) \leq B_2(M) < c_H$ for every $\mu \leq M$. The zero-cost rival prepares after every price, so Lemma A.2(a) gives $e \geq \rho$ in every candidate of each economy.

Step 1: the weak incumbent. Fix any candidate at F_0. By Lemma A.3(a) and the upper bounds in (A.6), each correctly signed order earns at most $\Delta_0 - k < 0$, and each wrong-signed order earns at most $-k$. Abstention is therefore the unique maximizer of U_H and of U_L, and (E5) gives $\sigma_H = \sigma_L = \delta_0$

under both tie rules for the informed trader. Then $f_H(x) = f_L(x) = (1-\lambda)\eta(x) + \lambda \mathbf{1}\{x = 0\}$, so (E1) gives $\mu(x) = \pi$ at every order, and (E2) gives $\bar{\mu}(\bar{P}) = \pi$ at every price. Since $B_0(\pi) < c_H$, (E3) gives $\zeta(\bar{P}) = 0$ with no tie, so $e(x) = \rho$. Condition (E4) and (4) then fix one price for every order. Every equilibrium at F_0 is therefore the tuple

$$\mathcal{E}_{\mathrm{N}}(F_0): \quad \begin{aligned} &\sigma_H = \sigma_L = \delta_0, \quad \mu(x) = \pi, \quad P(x) = \bar{P}_0 := t_{0,0} + \rho\,[t_{L,0} - t_{0,0} + \Delta_0 \pi], \\ &\bar{\mu}(\bar{P}_0) = \pi, \quad \zeta(\bar{P}_0) = 0, \end{aligned}$$

with truthful bids.

Equilibrium conditions for $\mathcal{E}_{\mathrm{N}}(F_0)$. (E1): $f_H = f_L$, so the posterior at every order is π. (E2): the single price pools all orders, and the odds of $\bar{\mu}(\bar{P}_0)$ are $\pi \sum_x f_H(x)/[(1-\pi)\sum_x f_L(x)] = \pi/(1-\pi)$. (E3): $B_0(\pi) < c_H$, strictly. (E4): by (A.4) with $e \equiv \rho$, $\mathbb{E}[V_T \mid X = x] = t_{0,0} + \rho\, w_0(\pi) = \bar{P}_0$. (E5): at $\mathcal{E}_{\mathrm{N}}(F_0)$, $K_H = \rho\Delta_0(1-\pi) < k$ and $K_L = \rho\Delta_0\pi < k$, so $U_H(1) < 0 = U_H(0)$ and $U_L(-1) < 0 = U_L(0)$, and the wrong-signed payoffs are at most $-k$. (E6): Appendix A.1. Entry is ρ, and high-class ownership is $\pi\rho W_H(F_0)$.

Step 2: the strong incumbent, orders in every candidate. Fix any candidate at F_1. By (A.6), $K_H \geq \rho\kappa\Delta_1 > k$ and $K_L \geq \rho\kappa\Delta_1 > k$. By Lemma A.3(a)–(b), the purchase is the unique maximizer of class H, and the sale is the unique maximizer of class L. Every equilibrium at F_1 has $\sigma_H = \delta_1$ and $\sigma_L = \delta_{-1}$, under both tie rules for the informed trader. These bounds use only the nonzero orders, so they hold for every $\beta < 1/2$.

Step 3: the strong incumbent, the tuple. Lemma A.1(d) gives the posteriors (m, π, M) after the orders $(-1, 0, 1)$. Step 0 gives $\Delta_1 > 0$ and $g_{H,1} > g_{L,1}$, so Lemma A.2 applies. Since $B_1(M) > c_H$, $B_1(\pi) < c_H$ and $B_1(m) < B_1(\pi)$, Lemma A.2(d) gives $e(1) = 1$ and $e(0) = e(-1) = \rho$, with no tie. Every equilibrium at F_1 is therefore the tuple

$$\mathcal{E}_{\mathrm{F}}(F_1): \quad \begin{aligned} &\sigma_H = \delta_1, \quad \sigma_L = \delta_{-1}, \quad (\mu(-1), \mu(0), \mu(1)) = (m, \pi, M), \\ &P(x) = t_{0,1} + e(x)\, w_1(\mu(x)), \quad \bar{\mu}(P(x)) = \mu(x), \quad \zeta(P(x)) = \mathbf{1}\{x = 1\}, \end{aligned}$$

with $(e(-1), e(0), e(1)) = (\rho, \rho, 1)$ and truthful bids.

Equilibrium conditions for $\mathcal{E}_{\mathrm{F}}(F_1)$. (E1): the likelihoods $f_H = (b, b_0, b+\lambda)$ and $f_L = (b+\lambda, b_0, b)$ give (m, π, M). (E2): the stated entry is nondecreasing in the posterior and lies in $[\rho, 1]$, and $\Delta_1 > 0$. By the paragraph on constructed tuples in Appendix A.2, the three prices are strictly increasing in the posterior, so $P(-1) < P(0) < P(1)$. Each price has one order in its preimage, so $\bar{\mu}(P(x)) = \mu(x)$. (E3): $B_1(M) > c_H$, $B_1(\pi) < c_H$ and $B_1(m) < c_H$, all strict. (E4): (A.4) with the stated entry. (E5): at $\mathcal{E}_{\mathrm{F}}(F_1)$, $K_H = \Delta_1(1-M) \geq \rho\kappa\Delta_1 > k$ and $K_L = \rho\Delta_1 m \geq \rho\kappa\Delta_1 > k$. Hence $U_H(1) > 0 = U_H(0) > U_H(-1)$ and $U_L(-1) > 0 = U_L(0) > U_L(1)$. (E6): Appendix A.1.

Step 4: entry, ownership and the takeover share at F_1. Class H places the order 1 with probability

$f_H(1) = b + \lambda$, through a liquidity purchase or an informed purchase. Class L places it with probability $f_L(1) = b$. Only the order 1 recruits the high-cost rival. Hence

$$e_H = \rho+(1-\rho)(b+\lambda), \qquad e_L = \rho+(1-\rho)b, \qquad \mathsf{E}_1 = \pi e_H+(1-\pi)e_L = \rho+(1-\rho)(b+\pi\lambda) > \rho,$$

since $b > 0$; also $e_H - e_L = (1-\rho)\lambda > 0$. Preparation depends on the order and the cost, and both are independent of (θ, R) given the class. A prepared rival of class j acquires the target when $\theta > R$, with probability $W_j(F_1)$. Write $\mathsf{O}_{j,i}$ for the probability that the rival is of class j and acquires the target at F_i, with $\pi_H = \pi$ and $\pi_L = 1 - \pi$. Then $\mathsf{O}_{j,1} = \pi_j e_j W_j(F_1)$, and Step 1 gives $\mathsf{O}_{j,0} = \pi_j \rho\, W_j(F_0)$. Division by $\pi > 0$ shows that high-class ownership rises exactly when $[\rho + (1-\rho)(b+\lambda)]\, W_H(F_1) > \rho\, W_H(F_0)$.

The rival acquires the target with positive probability against both incumbents. By (H1), $B_1(M) > c_H > 0$, so $g_{j,1} > 0$ for some class j; a class whose gross profit is positive beats the incumbent with positive probability, so $W_j(F_1) > 0$. Since F_1 strengthens F_0, $F_1 \le F_0$ pointwise, and $W_j(F_0) \ge W_j(F_1)$ for both classes. Class H's takeover share at F_i is $\mathsf{O}_{H,i}/(\mathsf{O}_{H,i} + \mathsf{O}_{L,i})$. For $x, y, x', y' \ge 0$ with $x + y > 0$ and $x' + y' > 0$, the inequality $x/(x+y) > x'/(x'+y')$ holds exactly when $xy' > x'y$. With $x = \mathsf{O}_{H,1}$, $y = \mathsf{O}_{L,1}$, $x' = \mathsf{O}_{H,0}$ and $y' = \mathsf{O}_{L,0}$, and after division by $\pi(1-\pi)\rho > 0$, the share rises from F_0 to F_1 exactly when

$$e_H\, W_H(F_1)\, W_L(F_0) > e_L\, W_H(F_0)\, W_L(F_1).$$

Let class H outbid F_1. Then $1 \ge W_H(F_0) \ge W_H(F_1) = 1$, so class H outbids F_0, and $\mathsf{O}_{H,1} = \pi e_H > \pi\rho = \mathsf{O}_{H,0}$. Condition (6) reads $e_H\, W_L(F_0) > e_L\, W_L(F_1)$. If $W_L(F_0) > 0$, then $e_H\, W_L(F_0) > e_L\, W_L(F_0) \ge e_L\, W_L(F_1)$, and the share rises strictly. If $W_L(F_0) = 0$, then $W_L(F_1) = 0$, so $\mathsf{O}_{L,0} = \mathsf{O}_{L,1} = 0$, and the share equals one against both incumbents.

Step 5: the very strong incumbent. Fix any candidate at F_2. By (A.6), $K_H \ge \rho\kappa\Delta_2 > k$ and $K_L \ge \rho\kappa\Delta_2 > k$, so every equilibrium has full orders, and Lemma A.1(d) gives the posteriors (m, π, M). By Step 0, $B_2(\mu) \le B_2(M) < c_H$ at each posterior, and every price belief is one of these posteriors or an average of them. So (E3) gives $\zeta \equiv 0$ with no tie, and $e \equiv \rho$. Every equilibrium at F_2 is therefore the tuple

$$\mathcal{E}_{\mathrm{F}}(F_2): \quad \begin{aligned} &\sigma_H = \delta_1, \quad \sigma_L = \delta_{-1}, \quad (\mu(-1), \mu(0), \mu(1)) = (m, \pi, M), \\ &P(x) = t_{0,2} + \rho\, w_2(\mu(x)), \quad \bar{\mu}(P(x)) = \mu(x), \quad \zeta \equiv 0, \end{aligned}$$

with truthful bids. *Equilibrium conditions.* (E1): the likelihoods $f_H = (b, b_0, b+\lambda)$ and $f_L = (b+\lambda, b_0, b)$ give (m, π, M). (E2): since $\Delta_2 > 0$ and $m < \pi < M$, the three prices $t_{0,2} + \rho\, w_2(\mu(x))$ differ, so each price has one order in its preimage. (E3): $B_2(\mu(x)) < c_H$ at each

posterior. (E4): (A.4) with $e \equiv \rho$. (E5): at $\mathcal{E}_{\mathrm{F}}(F_2)$, $K_H = \rho\Delta_2(1-M) > k$ and $K_L = \rho\Delta_2 m > k$. (E6): Appendix A.1. The prices differ, so the price stays informative while entry equals ρ. Entry is ρ after every order, so, as in Step 4, high-class ownership is $\pi\rho W_H(F_2)$.

Step 6: tie rules. Every comparison of the informed trader in Steps 1, 2 and 5 is strict. Every high-cost comparison at an equilibrium belief is strict. No tie rule is used, so the four pairs of rules give the same unique outcome in each economy. □

Lemma A.6 (Decisive advantage)**.** *Fix ranked finite-mean classes, a reserve $p > 0$, a high cost $c_H > 0$, and a family of continuous incumbent distributions. The family has* decisive advantage *if, for every distribution F in it,*

$$g_H(F) > c_H \quad \Longrightarrow \quad W_H(F) = 1.$$

At every distribution in this family with $\Delta_T > 0$ and $g_H > g_L$, any equilibrium that recruits a high-cost rival after some order has $W_H(F) = 1$. This holds at every interior prior, under all four tie rules, both at fixed presence and with research at a cap below one. Under (H1)–(H3), decisive advantage at F_1 gives the ownership and takeover-share conclusions under outbidding in Proposition 2(ii).

For point classes $h > \ell \geq p$ and the full family $F_r = U[0, r]$, $r > 0$, decisive advantage holds exactly when

$$c_H \geq \frac{h^2 - p^2}{2h}.$$

Proof. Full liquidity support and an interior prior give every price an interior posterior, at fixed presence and also when research is bounded by a cap below one. If a high-cost rival prepares after an order, then

$$c_H \leq B_F(\mu) = g_L(F) + \mu[g_H(F) - g_L(F)] < g_H(F),$$

where the last inequality uses $\mu < 1$ and the positive profit gap. Decisive advantage therefore gives $W_H(F) = 1$. Applied at F_1, this is the outbidding hypothesis of Proposition 2(ii).

For a point value h and $r > h$, direct integration gives

$$W_H(U[0,r]) = \frac{h}{r} < 1, \qquad g_H(U[0,r]) = \frac{h^2 - p^2}{2r}.$$

The latter decreases strictly in r, with supremum $(h^2 - p^2)/(2h)$ as $r \downarrow h$. If c_H is at least this supremum, $g_H > c_H$ forces $r \leq h$, where $W_H = 1$. If c_H is below it, continuity gives some $r > h$ with $g_H > c_H$ and $W_H < 1$. These two implications prove the equivalence for the full family. □

Each hypothesis is used once.

- $B_0(\pi) < c_H$ gives entry ρ at F_0. Through Proposition 1(a), it also stops recruitment after the zero order at F_1.
- $c_H < B_1(M)$ gives recruitment after a purchase at F_1.
- $\Delta_0 < k$ defeats every informed order at F_0, against every candidate.
- $k < \rho\kappa\Delta_1$ forces full orders at F_1, and through Proposition 1(b) at F_2, against every candidate.
- $B_2(M) < c_H$ stops recruitment at F_2.

The nonzero-order bounds hold for every $0 < \beta < 1/2$.

The open set. For uniform incumbents, (H1) to (H3) are strict inequalities between continuous functions of $(r_0, r_1, r_2, c_H, k, \rho, \pi, \lambda, \beta)$. They define an open set in the domain $0 < \beta < 1/2$. The sufficient condition $(1+\rho)W_H(F_1) \geq 2\rho W_H(F_0) > 0$ for higher ownership implies the exact condition of Step 4 whenever $b + \lambda > 1/2$, because then $e_H > (1+\rho)/2$.

The example

Corollary 1 gives the section of the open set through the example. In the example, $b + \lambda = 0.531737$.

Proposition 2 needs $B_0(\pi) < c_H$ in (H1); with $B_1(\pi) < c_H$ in its place, both cost types can prepare against the weaker incumbent: keep the example's other primitives, but set $c_H = 47/10$ and $r_2 = 8$. Then $k - \Delta_0 = 0.003333$, $c_H - B_1(1/2) = 0.408333$, $B_1(M) - c_H = 1.517155$, $\rho\kappa\Delta_1 - k = 0.024824$ and $c_H - B_2(M) = 0.312465$ are positive, but $c_H - B_0(1/2) = -0.104167$. At r_0, Step 1 gives zero orders and the posterior $1/2$, and $B_0(1/2) = 1153/240 > c_H$. Both cost types prepare, so entry is 1.000000 at r_0. Entry then falls to 0.522757 at r_1. Along the example's path the left side of (H2) also guards a conclusion. At $r = 1.55$ it fails, since $\Delta_T = 121/1240 > k$ there, and Proposition 3 gives two equilibria.

The tuples in the example. Table 3 lists $\mathcal{E}_{\mathrm{N}}(F_0)$, $\mathcal{E}_{\mathrm{F}}(F_1)$ and $\mathcal{E}_{\mathrm{F}}(F_2)$ in the example and the prior $\pi = 1/2$. In $\mathcal{E}_{\mathrm{F}}(F_1)$ the posteriors are $(0.268941,\ \pi,\ 0.731059)$ and the prices are 0.576074, 0.614583 and 1.362372. In $\mathcal{E}_{\mathrm{F}}(F_2)$ the prices are 0.610002, 0.664236 and 0.718471, and in $\mathcal{E}_{\mathrm{N}}(F_0)$ every order has the price 0.392708.

Corollary 1 (Recruitment and ownership in the example, full statement). *Let the classes be point masses at $h = 10$ and $\ell = 1$, the incumbent be uniform on $[0, r]$, and*

$$(p, \rho, c_H, k) = (1/2, 1/4, 6, 1/50), \qquad (r_0, r_1, r_2) = (6/5, 3, 18/5).$$

Take a slot with $0 < \lambda < 1$, $0 < \beta < 1/2$, $u = \lambda/[(1-\lambda)\beta]$.

(a) At prior $\pi = 1/2$, the high-cost window, trading-cost window and exclusion condition hold together exactly on

$$82/59 < u < 2686/1559,$$
$$\textit{equivalently} \quad \tau(r_1) = 141/200 < M(\pi) < 4245/5804 = \tau(r_2).$$

This is an interval of presence probabilities at each fixed liquidity share. Under all four tie rules, entry is ρ, $\rho + (1-\rho)(b + \lambda/2)$, ρ at the three strengths, and high-class ownership is $\rho/2$, $[\rho + (1-\rho)(b+\lambda)]/2$, $\rho/2$.

(b) For $u < 82/59$, entry at r_1 is ρ. At $u = 82/59$, orders are uniquely zero at r_0 and uniquely full at r_1, r_2. When an indifferent high-cost rival prepares, every conclusion of Proposition 2 holds. When it may prepare with any probability, entry after a purchase at r_1 ranges over $[\rho, 1]$, and total entry there ranges over $[\rho, \rho + (1-\rho)(b + \lambda/2)]$. For $2686/1559 \le u < 19/3$, a purchase at r_2 induces certain preparation when an indifferent high-cost rival prepares. For $u \ge 19/3$, the strict upper inequality of the trading-cost window at r_1 fails.

(c) If $1 + u = \mathrm{e}$, all three hypotheses hold exactly when

$$\frac{\tau(r_1)}{\tau(r_1) + [1 - \tau(r_1)]\mathrm{e}} < \pi < \frac{\tau(r_2)}{\tau(r_2) + [1 - \tau(r_2)]\mathrm{e}},$$

with endpoints 0.467850 and 0.500424 to six places. The high-cost and trading-cost windows alone hold exactly between the same lower endpoint and $\tau(r_0) = 195/308$.

(d) At $\beta = 0.294655$, $1 + u = \mathrm{e}$ and $\pi = 1/2$, entry at the three strengths is 0.250000, 0.522757, 0.250000, and high-class ownership is 0.125000, 0.324401, 0.125000. Class H's share of acquisitions won by the rival is respectively 0.545455, 0.830691, 0.782609.

(e) For these point values and the strength range $[r_0, r_2]$, decisive advantage on the full uniform family and the affordability condition $h - r_2 \ge c_H$ of Proposition 3(c) hold together exactly on the cost window

$$399/80 \le c_H \le 32/5.$$

The example's cost $c_H = 6$ lies in this window, with margins $81/80$ and $2/5$. For these class values, reserve and cost, every economy with a uniform incumbent $U[0, r]$ in which some order recruits a high-cost rival has

$$r < h - c_H + \sqrt{(h - c_H)^2 - p^2} \approx 7.968627 < h,$$

whatever the slot, prior, ρ, k and research.

Proof of Corollary 1. Take the example's primitives $(h, \ell, p) = (10, 1, 1/2)$, $(\rho, c_H, k) = (1/4, 6, 1/50)$ and the strengths $(r_0, r_1, r_2) = (6/5, 3, 18/5)$. For $\ell \le r \le h$, $\tau(r) = [2rc_H - (\ell^2 - p^2)]/(2rh - r^2 - \ell^2)$, with $\tau(r_0) = 195/308$, $\tau(r_1) = 141/200$ and $\tau(r_2) = 4245/5804$. Also $\Delta_T(r_0) = 1/60$, $\Delta_T(r_1) = 2/3$ and $\Delta_T(r_2) = 169/180$ by (8).

Part (a). At the prior $\pi = 1/2$, $M = (1+u)/(2+u)$ and $\kappa = m = 1 - M = 1/(2+u)$. The left side of (H1) has the margin $c_H - B_{r_0}(1/2) = 1.195833$, and the left side of (H2) has the margin $k - \Delta_T(r_0) = 0.003333$. Neither depends on the slot. Since

$$M > \tau \quad \Longleftrightarrow \quad u > \frac{2\tau - 1}{1 - \tau},$$

the right side of (H1) is $u > 82/59$, and (H3) is $u < 2686/1559$. The right side of (H2) is

$$\rho(1 - M)\Delta_T(r_1) > k \quad \Longleftrightarrow \quad u < \frac{\rho\Delta_T(r_1)}{k} - 2 = 19/3.$$

The bound $u < 2686/1559$ implies it; at the example's slot $19/3 - u = 4.615052$. Hence (H1) to (H3) hold if and only if $82/59 < u < 2686/1559$, that is $\tau(r_1) < M < \tau(r_2)$. At fixed β, $\lambda = u\beta/(1 + u\beta)$ increases strictly in u, so the band is an open interval of λ. Proposition 2(i)–(iii) then give the entry values under all four tie rules, and since $h > r_2$, $W_H = 1$ at the three strengths, which gives the ownership values.

Part (b). For $u < 19/3$, Step 2 of the proof of Proposition 2 holds at r_1, and at r_2 because $\Delta_T(r_2) > \Delta_T(r_1)$. Orders are full at both strengths, and the posteriors are $(m, 1/2, M)$. If $u < 82/59$, then $M < \tau(r_1)$, no order recruits at r_1, and entry at r_1 is ρ. If $2686/1559 \le u < 19/3$, then $M \ge \tau(r_2)$, and a purchase recruits at r_2 when an indifferent high-cost rival prepares. For $u \ge 19/3$, the right side of (H2) fails at r_1.

At $u = 82/59$, the left side of (H1), both sides of (H2) and (H3) still hold strictly. At r_0, every correct order earns at most $\Delta_T(r_0) - k = -0.003333 < 0$, so orders are uniquely zero. At r_1 and r_2, orders are uniquely full by Step 2, and parts (i) and (iii) hold. A purchase at r_1 leaves the high-cost rival indifferent, and an indifferent high-cost rival prepares and gives entry E_1. When an indifferent high-cost rival may prepare with any probability, entry after a purchase ranges over $[\rho, 1]$, so entry at r_1 can take any value in $[\rho, \mathsf{E}_1]$.

Part (c). Fix $1 + u = \mathrm{e}$ and the example's other inputs, and vary π. The inequality $M(\pi) > \tau$ is $\pi/(1-\pi) > \tau/[(1-\tau)(1+u)]$, that is $\pi > \tau/[\tau + (1-\tau)\mathrm{e}]$. The right side of (H1) gives the lower endpoint of the corollary, and (H3) gives its upper endpoint. The left side of (H1) is $B_{r_0}(\pi) < c_H$, that is $\pi < \tau(r_0) = 195/308$. The left side of (H2) does not depend on π. For the right side of (H2),

let $\kappa_0 = k/[\rho\Delta_T(r_1)] = 3/25$. Then

$$m(\pi) > \kappa_0 \quad \Longleftrightarrow \quad \pi > \frac{\kappa_0 \mathrm{e}}{1 - \kappa_0 + \kappa_0 \mathrm{e}}.$$

This gives the lower endpoint, and $1 - M(\pi) > \kappa_0$ gives the upper endpoint, so $\pi \in (0.270432, 0.729568)$ approximately. This interval contains the other two. The intersection is the interval of the corollary, with endpoints ≈ 0.467850 and ≈ 0.500424. Without (H3), the upper endpoint is $\tau(r_0) \approx 0.633117$.

Part (d). At the stated slot, $b = \beta/[1 + (\mathrm{e} - 1)\beta]$ and $\lambda = (\mathrm{e} - 1)b$. Substitution in part (a) gives the entry and ownership values. Since $h > r_2$, $W_H = 1$ at every stated strength and $W_L = \ell/r$. Thus the takeover share is

$$\frac{\pi e_H}{\pi e_H + (1 - \pi)e_L \ell/r}.$$

At r_0 and r_2, both class entries equal ρ; at r_1, they are the two expressions in Proposition 2(ii). These substitutions give the three displayed shares.

Part (e). Lemma A.6 makes decisive advantage equivalent to $c_H \geq (h^2 - p^2)/(2h) = 399/80$; affordability is $c_H \leq h - r_2 = 32/5$. Their intersection is the displayed cost window. The distances from the stated cost to the two endpoints give the displayed margins. On $p \leq r \leq h$, $g_H(r) = h - r/2 - p^2/(2r)$ decreases strictly for $r > p$. Solving $g_H(r) = c_H$ gives the upper root in the statement; the lower algebraic root lies below p. For $r \leq p$, gross profit is $h - p > c_H$, and for $r \geq h$, it is at most $(h^2 - p^2)/(2h) < c_H$. Hence this upper root is the unique actual crossing. Recruitment implies $g_H > c_H$ by Lemma A.6's first argument, which gives the strict bound on r. □

The recruitment region. Table 2 gives the region in the example. The trading bound $\kappa(\pi) > k/[\rho\Delta_T(r_1)]$ at $k = 1/50$ holds on $(0.270432, 0.729568)$. Some trading cost satisfies (H2) exactly when $\Delta_T(r_0) < \rho\kappa\Delta_T(r_1)$, or $\kappa(\pi) > \Delta_T(r_0)/[\rho\Delta_T(r_1)]$, on $(0.231969, 0.768031)$. Table A.6 gives the margins of the hypotheses and the example's distance to each boundary of the region.

A.4 Coexistence

This subsection states in full and proves Proposition 3, Corollary 2 and Corollary A.2. Its first part holds against every continuous incumbent distribution with a positive spread and $g_H > g_L$, and along every path of such distributions; its second part applies the first to the example.

General results

Setting. Throughout this part the classes satisfy the class ranking, and the prior $\pi \in (0, 1)$, the slot (λ, β) with $0 < \lambda < 1$ and $0 < \beta < 1/2$, and $\rho \in (0, 1)$, $c_H > 0$ and $k > 0$ are fixed. An incumbent

distribution F is continuous, with $\Delta_T(F) > 0$ and $g_H(F) > g_L(F)$. Against F write Δ_T, τ and $J = \Delta_T(1-\tau)$ for the spread, the preparation threshold and the gross return of a purchase that just reaches τ and recruits. Write $M = M(\pi)$, $m = m(\pi)$,

$$\Delta_T^{\mathrm{F}} = \frac{k}{1-M}, \qquad \Delta_T^{\mathrm{N}} = \frac{k}{\rho \max\{\pi, 1-\pi\}}.$$

By Lemma A.3(a), no equilibrium puts weight on a wrong-signed order. Write $s_+ = \lambda\sigma_H(1)$ and $s_- = \lambda\sigma_L(-1)$. On the orders $(-1, 0, 1)$, $f_H = (b,\ b_0+\lambda-s_+,\ b+s_+)$ and $f_L = (b+s_-,\ b_0+\lambda-s_-,\ b)$, so

$$\mu_+(s_+) = \frac{\pi(b+s_+)}{b+\pi s_+}, \qquad 1-\mu_+(s_+) = \frac{(1-\pi)\,b}{b+\pi s_+}, \qquad \mu_-(s_-) = \frac{\pi b}{b+(1-\pi)s_-},$$

and $\mu_0(s_+, s_-)$ has the odds $\frac{\pi}{1-\pi} \cdot \frac{b_0+\lambda-s_+}{b_0+\lambda-s_-}$. On $[0, \lambda]$, μ_+ rises strictly from π to M, and μ_- falls strictly from π to m. The same formula defines $\mu_+(s)$ for every $s \geq 0$, rising strictly to one. For $\pi < \tau < 1$ let

$$s^* = \frac{b\,(\tau-\pi)}{\pi\,(1-\tau)} > 0,$$

the rate at which a purchase reaches the threshold, $\mu_+(s^*) = \tau$. Then $\mu_+(s_+) < \tau$ exactly when $s_+ < s^*$; the rate s^* is feasible, $s^* \leq \lambda$, exactly when $\tau \leq M$, and $s^* < \lambda$ exactly when $\tau < M$. If $\tau \geq M$, and in particular if $\tau \geq 1$, then $\mu_+(s_+) < \tau$ for every $s_+ < \lambda$.

A path is a family $(F_r)_{r\in I}$ of such distributions over an interval I of strengths, with $F_{r'}$ first-order stochastically dominating F_r for $r' > r$ and every other primitive fixed. Along a path $\Delta_T(r)$ is nondecreasing, $g_H(r)$, $g_L(r)$ and $\gamma(r) = g_H(r) - g_L(r)$ are nonincreasing, and $\Delta_T(r) + \gamma(r)$ is constant (Lemma A.5). A set of strengths is an up-set if it contains every strength above each of its members, and a down-set if it contains every strength below each of its members.

The posteriors follow from (E1), because the low class never buys and class H never sells. The posterior $\mu_+(s)$ is continuous in $s \geq 0$, with derivative of the sign of $\pi b(1-\pi) > 0$, value π at $s = 0$, value M at $s = \lambda$ and limit one, and the equation $\mu_+(s) = \tau$ is the linear equation $\pi(b+s) = \tau(b+\pi s)$, whose unique root is s^*. The statements on s^* follow. Entry after each order follows from Lemma A.2(d) at its posterior.

Lemma A.7 (The high class)**.** *Let $\pi < \tau$. In a candidate with no wrong-signed order, the pair $(s_+, e(1))$ satisfies (E1) to (E3) at the order 1, and (E5) for class H, exactly when it is one of these five pairs.*

(N) $s_+ = 0$ *and* $e(1) = \rho$, *with* $\rho\Delta_T(1-\pi) \leq k$.

(Q) $0 < s_+ < \lambda$, $\mu_+(s_+) < \tau$, $e(1) = \rho$ *and* $\rho\Delta_T[1-\mu_+(s_+)] = k$. *This pair needs an indifferent informed trader who may mix.*

(R) $0 < s_+ < \lambda$, $\mu_+(s_+) > \tau$, $e(1) = 1$ *and* $\Delta_T[1 - \mu_+(s_+)] = k$. *This pair needs an indifferent informed trader who may mix.*

(T) $0 < s_+ < \lambda$, $\mu_+(s_+) = \tau$, *that is* $s_+ = s^*$, *and* $e(1) = e^* := k/J$. *When an indifferent high-cost rival prepares, this requires* $e^* = 1$, *that is* $J = k$. *When an indifferent high-cost rival may prepare with any probability, it requires* $\rho \le e^* \le 1$, *that is* $\rho J \le k \le J$. *This pair needs an indifferent informed trader who may mix.*

(F) $s_+ = \lambda$. *Here* $e(1) = 1$ *if* $M > \tau$ *and* $e(1) = \rho$ *if* $M < \tau$. *If* $M = \tau$, *then* $e(1) = 1$ *when an indifferent high-cost rival prepares, and* $e(1) \in [\rho, 1]$ *when it may prepare with any probability. The return satisfies* $e(1)\Delta_T(1 - M) \ge k$ *when an indifferent informed trader may mix, and* $e(1)\Delta_T(1 - M) > k$ *when an indifferent informed trader abstains.*

Each equation in (Q) and (R) has at most one solution. If $\tau \ge M$, *and in particular if* $\tau \ge 1$, *the pairs (R) and (T) do not occur, and (F) has* $e(1) = \rho$ *when* $\tau > M$.

Proof. By Lemma A.3(b), (E5) for class H requires $K_H \le k$ at $s_+ = 0$. At $s_+ = \lambda$ it requires $K_H \ge k$ when an indifferent informed trader may mix, and $K_H > k$ when an indifferent informed trader abstains, because that rule selects abstention at a tie. At $0 < s_+ < \lambda$ it requires $K_H = k$; an interior rate is a mixture at indifference, so it needs an indifferent informed trader who may mix. Here $K_H = e(1)\Delta_T[1 - \mu_+(s_+)]$, and $e(1)$ follows from Lemma A.2(d) at the posterior $\mu_+(s_+)$.

- At $s_+ = 0$, $\mu_+(0) = \pi < \tau$, so $e(1) = \rho$. This is (N).
- At $0 < s_+ < \lambda$ with $\mu_+(s_+) < \tau$: $e(1) = \rho$, and $K_H = k$ is the equation of (Q).
- At $0 < s_+ < \lambda$ with $\mu_+(s_+) > \tau$: $e(1) = 1$, and $K_H = k$ is the equation of (R).
- At $0 < s_+ < \lambda$ with $\mu_+(s_+) = \tau$: then $\pi < \tau < M < 1$ and $s_+ = s^*$, so $J > 0$, and $K_H = e(1)\Delta_T(1 - \tau) = e(1)J = k$ gives $e(1) = e^*$. When an indifferent high-cost rival prepares, $e(1) = 1$; when it may prepare with any probability, $e(1)$ can take any value in $[\rho, 1]$. This is (T).
- At $s_+ = \lambda$, $\mu_+(\lambda) = M$. Lemma A.2(d) gives $e(1)$, and the condition on K_H is that of (F).

The function $1 - \mu_+$ falls strictly and $\Delta_T > 0$, so each equation has at most one root. If $\tau \ge M$, then $\mu_+(s_+) < M \le \tau$ for every $s_+ < \lambda$, so (R) and (T) do not occur, and $M < \tau$ gives $e(1) = \rho$ in (F). □

Lemma A.8 (The low class)**.** *Let* $\pi < \tau$. *Then* $\mu_-(s_-) \le \pi < \tau$ *for every* s_-, *so* $e(-1) = \rho$ *in every candidate. In a candidate with no wrong-signed order, the rate* s_- *satisfies (E1) to (E3) at the order* -1, *and (E5) for class L, exactly when it is one of these:*

(N) $s_- = 0$, *with* $\rho\Delta_T\pi \le k$;

(Q) $0 < s_- < \lambda$, *with* $\rho\Delta_T\mu_-(s_-) = k$; *this rate needs an indifferent informed trader who may mix;*

(F) $s_- = \lambda$, *with* $\rho\Delta_T m \ge k$ *when an indifferent informed trader may mix, and* $\rho\Delta_T m > k$ *when an indifferent informed trader abstains.*

When an indifferent informed trader may mix, exactly one rate qualifies:

$$\hat{s}_- = \begin{cases} 0, & \rho\Delta_T\pi \le k, \\ \dfrac{b\,(\pi\rho\Delta_T/k - 1)}{1-\pi}, & \rho\Delta_T m < k < \rho\Delta_T\pi, \\ \lambda, & \rho\Delta_T m \ge k. \end{cases}$$

When an indifferent informed trader abstains, no rate qualifies if $\rho\Delta_T m \le k < \rho\Delta_T\pi$.

Proof. Since μ_- is decreasing, $\mu_-(s_-) \le \mu_-(0) = \pi < \tau$, so Lemma A.2(d) gives $e(-1) = \rho$. Then $K_L = \rho\Delta_T\mu_-(s_-)$ is continuous and strictly decreasing in s_-, from $\rho\Delta_T\pi$ at $s_- = 0$ to $\rho\Delta_T m$ at $s_- = \lambda$. By Lemma A.3(b), (E5) for class L requires $K_L \le k$ at $s_- = 0$; $K_L \ge k$ at $s_- = \lambda$ when an indifferent informed trader may mix, and $K_L > k$ when it abstains; and $K_L = k$ at $0 < s_- < \lambda$, which needs an indifferent informed trader who may mix. If $\rho\Delta_T\pi \le k$, then $K_L < k$ for every $s_- > 0$, so only $s_- = 0$ qualifies. If $\rho\Delta_T m \ge k$, then $K_L > k$ for every $s_- < \lambda$, so only $s_- = \lambda$ qualifies. Otherwise $\rho\Delta_T m < k < \rho\Delta_T\pi$, and the intermediate value theorem and strict monotonicity give exactly one interior root. The equation $\pi b/[b + (1-\pi)s_-] = k/(\rho\Delta_T)$ gives the formula. When an indifferent informed trader abstains and $\rho\Delta_T m \le k < \rho\Delta_T\pi$, the rate $s_- = 0$ fails because $K_L > k$ there; the rate $s_- = \lambda$ fails because $K_L \le k$ there; and an interior rate needs an indifferent informed trader who may mix. □

Lemma A.9 (Product structure)**.** *Let* $\pi < \tau$, *and take rates with* $s_+ \ge s_-$. *Then* $\mu_0(s_+, s_-) \le \pi < \tau$. *Under each pair of tie rules, the tuple with the rates* (s_+, s_-) *is an equilibrium if and only if these four conditions hold:*

(i) $(s_+, e(1))$ *is a pair of Lemma A.7;*

(ii) s_- *is a rate of Lemma A.8;*

(iii) $e(0) = e(-1) = \rho$;

(iv) the prices are $P(x) = t_0 + e(x)\,w(\mu(x))$, *and* $\bar{\mu}(P(x)) = \mu(x)$.

Proof. If $s_+ \geq s_-$, then $b_0 + \lambda - s_+ \leq b_0 + \lambda - s_-$, so the odds of μ_0 are at most $\pi/(1-\pi)$. Hence $\mu_0 \leq \pi < \tau$.

Necessity. In any equilibrium, Lemma A.2(d) gives $e(0) = e(-1) = \rho$ at the posteriors μ_0 and $\mu_-(s_-)$, both below τ, and Lemma A.2(b)–(c) give condition (iv). Lemma A.3(d) shows that (E5) for class H involves only $(s_+, \mu(1), e(1))$, and (E5) for class L only $(s_-, \mu(-1), e(-1))$. So (E5) gives (i) and (ii).

Sufficiency. Take rates and entry that satisfy (i) to (iv). Each condition of Definition A.1 holds at this tuple. (E1): the posteriors are computed from f_H and f_L. (E2): the stated entry is ρ at the posteriors $\mu_-(s_-)$ and μ_0, both at most $\pi < \tau$, and it is $e(1) \geq \rho$ at $\mu_+(s_+) \geq \pi$, with $e(1) = 1$ when $\mu_+(s_+) > \tau$. So entry lies in $[\rho, 1]$ and is nondecreasing in the posterior. Two orders with a common posterior include the order 0 or -1, so the common posterior is at most π; if the order 1 is one of them, then $\mu_+(s_+) \leq \pi$ forces $s_+ = 0$, the pair (N), with $e(1) = \rho$. Orders with a common posterior therefore have a common entry. By the paragraph on constructed tuples in Appendix A.2, orders share a price exactly when they share the posterior and the entry, the preparation rule $\zeta(P(x)) = [e(x) - \rho]/(1-\rho)$ is a function of the price, and the belief at each price is the common posterior of its orders, which is (E2). (E3): at each price the belief is the posterior of its orders, and the stated entry is the one that Lemma A.2(d) allows at that posterior under the tie rule for the rival in use; condition (i) states it at the order 1, and the posteriors $\mu_-(s_-)$ and μ_0 lie below τ. (E4): (A.4) with the stated entry. (E5): by Lemma A.3(d), the payoffs of class H involve only $(s_+, \mu(1), e(1))$, and those of class L only $(s_-, \mu(-1), e(-1))$; conditions (i) and (ii) state optimality for each, and Lemma A.3(a) rules out the wrong-signed orders. (E6): Appendix A.1. □

Proposition 3 (Uninformative and informative markets, full statement). *Let the classes satisfy the class ranking, and fix the prior, the slot, ρ, c_H and k as above. Parts (a) to (c), except the second sentence of (c)(iv), take the default tie rules: an indifferent informed trader may mix over its best orders, and an indifferent high-cost rival prepares. That sentence and part (d) state their own.*

(a) The region. *An uninformative market and an informative market coexist against F if and only if*

$$\pi < \tau \leq M \qquad \text{and} \qquad \frac{k}{1-\tau} \leq \Delta_T \leq \Delta_T^{\mathrm{N}}. \tag{A.7}$$

Against such an F the equilibria are exactly the uninformative market and one informative market. In the informative market the low class abstains, class H buys at the rate $s_+ = \min\{\lambda,\ (b/\pi)\,[(1-\pi)\Delta_T/k - 1]\}$, a purchase recruits a high-cost rival, and entry after the other two orders is ρ. It is a full purchase if $\Delta_T \geq \Delta_T^{\mathrm{F}}$, the threshold pair with entry one after a purchase if $\Delta_T < \Delta_T^{\mathrm{F}}$ and $J = k$, and a mixed purchase that recruits if $\Delta_T < \Delta_T^{\mathrm{F}}$ and $J > k$. Its entry is

$$\mathsf{E} = \rho + (1-\rho)\min\left\{\frac{(1-\pi)\,b\,\Delta_T}{k},\ b + \pi\lambda\right\}, \tag{A.8}$$

which depends on F only through the spread, rises strictly with the spread on $\Delta_T < \Delta_T^{\mathrm{F}}$ and is constant on $\Delta_T \geq \Delta_T^{\mathrm{F}}$. The set of pairs (Δ_T, τ) that satisfy (A.7) is convex, and each of its points has $\tau \leq 1 - \rho \max\{\pi, 1-\pi\}$. Coexistence therefore requires $\pi < 1/(1+\rho)$.

(b) The sure-purchase stretch. *Along a path, the strengths at which an uninformative market coexists with a full purchase, those with $\pi < \tau(r) \leq M$ and $\Delta_T^{\mathrm{F}} \leq \Delta_T(r) \leq \Delta_T^{\mathrm{N}}$, form an interval. Every strength at which an uninformative market coexists with a mixed purchase that recruits, or with the threshold pair, lies below every strength of this interval. If an uninformative market and an informative market coexist at r' and at r'' but not at r, with $r' < r < r''$, then $\Delta_T(r) < \Delta_T^{\mathrm{F}}$: a gap lies at spreads too narrow for a sure purchase to pay.*

(c) The balance condition. *Along a path, let the product $\Delta_T(r)\,[g_H(r) - c_H]$ be nondecreasing in r.*

(i) The strengths with $J(r) \geq k$ form an up-set, on which J is nondecreasing and rises strictly wherever Δ_T rises strictly. The strengths at which an uninformative market and an informative market coexist form an interval, at every prior and every slot.

(ii) Let I be compact, let Δ_T and τ be continuous in r, and let $\tau > \pi$ at the weakest incumbent of the path. Then this interval is closed. When it is not empty, it starts at the weakest strength with $J(r) \geq k$ and ends at the smaller of the strongest strength with $\tau(r) \leq M$ and the strongest strength with $\Delta_T(r) \leq \Delta_T^{\mathrm{N}}$.

(iii) The product $\Delta_T(r)\,[g_H(r) - c_H]$ is nondecreasing, and J rises strictly, when the incumbent is uniform, $F_r = U[0, r]$ for r in a compact interval I with $p < \min I$, every low-class value lies in $[p, \min I]$, every high-class value is at least $\max I$, and $\mathbb{E}_H\theta \geq \max I + c_H$.

(iv) If, in addition, $\Delta_T(r)\,[g_H(r) - c_H]^2$ is nondecreasing, then $\Delta_T(r)\,[1 - \tau(r)]^2$ is nondecreasing. When an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, the threshold pair is a pair of Lemma A.7 exactly at the strengths with $\pi < \tau(r) < M$ and $\rho J(r) \leq k \leq J(r)$; on these strengths its entry, with entry ρ after a sale and after the zero order,

$$\mathsf{E}_{\mathrm{T}}(r) = \rho + (1-\pi)\, b\, \frac{k - \rho J(r)}{\Delta_T(r)\,[1 - \tau(r)]^2},$$

is nonincreasing. Where $\tau(r) = M$, the threshold rate is $s^* = \lambda$, and a purchase at that rate is a full purchase, the pair (F) of Lemma A.7.

(d) Existence at the prior $1/2$. *Let the prior be $1/2$ and $\tau > 1/2$, including $\tau \geq 1$, and let an indifferent informed trader mix over its best orders. Under either tie rule for the rival an*

equilibrium exists against F. If $\rho\Delta_T \le 2k$, the uninformative market is one. If $\rho\Delta_T > 2k$, the low-class rate of Lemma A.8 is $\hat{s}_- = \min\{\lambda,\ b\,(\rho\Delta_T/k - 2)\} > 0$. If $\mu_+(\hat{s}_-) < \tau$, both classes trade at the rate $\hat{s}_-$ and entry is ρ after every order. If $\mu_+(\hat{s}_-) \ge \tau$, the low class sells at the rate $\hat{s}_-$, class H buys at the rate $\min\{\lambda,\ b\,(\Delta_T/k - 2)\} \ge \hat{s}_-$, a purchase recruits a high-cost rival, and entry after a sale and after the zero order is ρ.

An uninformative market is an equilibrium with $\sigma_H = \sigma_L = \delta_0$, and an informative market is an equilibrium with informed trade.

Proof of Proposition 3(a). Fix F and the default tie rules. The proof has four steps.

Step 1: a prior at or above the threshold. Let $\pi \ge \tau$. In an uninformative market every posterior is $\pi \ge \tau$, so Lemma A.2(d) gives entry one after every order, and Lemma A.3 gives $K_H = \Delta_T(1-\pi)$ and $K_L = \Delta_T\pi$. An uninformative market therefore needs $\Delta_T \max\{\pi, 1-\pi\} \le k$. Under this inequality no candidate with informed trade is an equilibrium. By Lemma A.3(a) such a candidate puts no weight on a wrong-signed order. If $s_+ > 0$, then $\mu(1) = \mu_+(s_+) > \pi \ge \tau$, so $e(1) = 1$ and $K_H = \Delta_T[1 - \mu_+(s_+)] < \Delta_T(1-\pi) \le k$: class H strictly prefers abstention. If $s_- > 0$, then $\mu(-1) = \mu_-(s_-) < \pi$ and $e(-1) \le 1$, so $K_L < \Delta_T\pi \le k$: class L strictly prefers abstention. So coexistence needs $\pi < \tau$.

Step 2: the uninformative market when $\pi < \tau$. Let $\pi < \tau$, so Lemmas A.7 to A.9 apply. An uninformative market has $s_+ = s_- = 0$, so by Lemma A.9 it is an equilibrium exactly when $(0, e(1))$ is the pair (N) of Lemma A.7 and 0 is the rate (N) of Lemma A.8, that is when $\rho\Delta_T(1-\pi) \le k$ and $\rho\Delta_T\pi \le k$, or $\Delta_T \le \Delta_T^{\mathrm{N}}$. Its entry is then ρ after every order and its prices follow from condition (iv), so there is exactly one uninformative market.

Step 3: informative markets when $\pi < \tau$ and $\Delta_T \le \Delta_T^{\mathrm{N}}$. Since $\rho\Delta_T\pi \le k$, Lemma A.8 gives $s_- = 0$ in every equilibrium, so $s_+ \ge s_-$, and by Lemma A.9 the equilibria are the pairs of Lemma A.7 with $s_- = 0$ and entry ρ after the orders 0 and -1. An informative market has $s_+ > 0$. The pair (Q) is impossible, because $\rho\Delta_T[1 - \mu_+(s_+)] < \rho\Delta_T(1-\pi) \le k$ for $s_+ > 0$, and so is (F) with $M < \tau$, for the same reason. The remaining pairs, (R), (T) and (F) with $M \ge \tau$, need $\mu_+(s_+) \ge \tau$ for some $s_+ \le \lambda$, so $\tau \le M$; since $e(1) \le 1$, they also need $k \le e(1)\Delta_T[1 - \mu_+(s_+)] \le \Delta_T(1-\tau) = J$, that is $\Delta_T \ge k/(1-\tau)$. With Step 1, coexistence implies (A.7).

Conversely, let F satisfy (A.7). Then $\tau \le M < 1$ and $J \ge k$, and Step 2 gives the uninformative market. If $\Delta_T \ge \Delta_T^{\mathrm{F}}$, that is $\Delta_T(1-M) \ge k$, the pair (F) holds with $e(1) = 1$, which an indifferent high-cost rival chooses when $M = \tau$. It is the only informative pair: at every $s_+ < \lambda$ with $\mu_+(s_+) \ge \tau$ the return $\Delta_T[1 - \mu_+(s_+)]$ exceeds $\Delta_T(1-M) \ge k$, so neither (R) nor (T) holds. If $\Delta_T < \Delta_T^{\mathrm{F}}$, then $\Delta_T(1-M) < k \le J = \Delta_T(1-\tau)$, so $\tau < M$ and (F) fails. The return $s \mapsto \Delta_T[1 - \mu_+(s)]$ is continuous and strictly decreasing on $[s^*, \lambda]$, from $J \ge k$ to $\Delta_T(1-M) < k$, so it equals k at exactly one $s_+ \in [s^*, \lambda)$. At $s_+ = s^*$, which happens exactly when $J = k$, this is (T) with $e^* = 1$; at $s_+ > s^*$

it is (R); and no other informative pair exists. In every case the low class abstains, entry after the orders 0 and -1 is ρ, and Lemma A.9 completes the tuple. The equation $\Delta_T(1-\pi)b/(b+\pi s)=k$ has the root $s=(b/\pi)[(1-\pi)\Delta_T/k-1]$, which is at least λ exactly when $\Delta_T(1-M)\geq k$, because $1-M=(1-\pi)b/(b+\pi\lambda)$. This gives the stated rate.

Step 4: entry and the shape of the region. Entry is $\mathsf{E}=\sum_x \Pr(X=x)\,e(x)$. With $e(0)=e(-1)=\rho$, $e(1)=1$ and $\Pr(X=1)=\pi(b+s_+)+(1-\pi)b=b+\pi s_+$, entry is $\rho+(1-\rho)(b+\pi s_+)$, and $b+\pi s_+=\min\{(1-\pi)b\Delta_T/k,\ b+\pi\lambda\}$ gives (A.8). The prior, b, λ and k are fixed, so entry depends on F only through Δ_T. The two terms of the minimum are equal exactly at $\Delta_T=\Delta_T^{\mathrm{F}}$, so entry rises strictly with Δ_T below Δ_T^{F} and is constant above. With $\tau\leq M<1$, the inequality $k/(1-\tau)\leq\Delta_T$ is $\tau\leq 1-k/\Delta_T$. The set of pairs (Δ_T,τ) that satisfy (A.7) is therefore the intersection of the hypograph of the concave function $\Delta_T\mapsto 1-k/\Delta_T$ on $\Delta_T>0$ with three half-planes, and it is convex. Its two spread bounds give $1-\tau\geq\rho\max\{\pi,1-\pi\}$. So $\pi<\tau\leq 1-\rho\max\{\pi,1-\pi\}\leq 1-\rho\pi$, which requires $\pi<1/(1+\rho)$. □

Proof of Proposition 3(b). Along a path, $B_r(\mu)=g_L(r)+\mu\gamma(r)=\mu g_H(r)+(1-\mu)g_L(r)$ is nonincreasing in r at every $\mu\in[0,1]$, because g_H and g_L are, and Δ_T is nondecreasing (Lemma A.5). Since $\gamma>0$, the condition $\pi<\tau(r)$ is $B_r(\pi)<c_H$, an up-set, and $\tau(r)\leq M$ is $B_r(M)\geq c_H$, a down-set; $\Delta_T(r)\geq\Delta_T^{\mathrm{F}}$ is an up-set, and $\Delta_T(r)\leq\Delta_T^{\mathrm{N}}$ a down-set. An intersection of up-sets and down-sets is an interval: if $r'<r<r''$ and r' and r'' belong to it, then r belongs to each up-set because r' does, and to each down-set because r'' does.

By part (a), an uninformative market coexists with a full purchase exactly where (A.7) holds and $\Delta_T(r)\geq\Delta_T^{\mathrm{F}}$. There $J=\Delta_T(1-\tau)\geq\Delta_T(1-M)\geq k$, so the condition $k/(1-\tau)\leq\Delta_T$ holds, and the set is $\{r:\pi<\tau(r)\leq M,\ \Delta_T^{\mathrm{F}}\leq\Delta_T(r)\leq\Delta_T^{\mathrm{N}}\}$, an interval. By part (a), a strength r at which an uninformative market coexists with a mixed purchase that recruits or with the threshold pair has $\Delta_T(r)<\Delta_T^{\mathrm{F}}\leq\Delta_T(r')$ at every strength r' of this interval, so $r<r'$, because Δ_T is nondecreasing. Finally, $\{r:\pi<\tau(r)\leq M,\ \Delta_T(r)\leq\Delta_T^{\mathrm{N}}\}$ is an interval that contains every strength with coexistence. If coexistence holds at r' and r'' but not at $r\in(r',r'')$, then r lies in this interval, and $\Delta_T(r)\geq\Delta_T^{\mathrm{F}}$ would put r in the set of full purchases; so $\Delta_T(r)<\Delta_T^{\mathrm{F}}$. □

Proof of Proposition 3(c)(i) and (c)(ii). Since $1-\tau=(g_H-c_H)/\gamma$, $J=\Delta_T\,(g_H-c_H)/\gamma$ at every strength, with $\gamma>0$ nonincreasing and $\gamma=\bar{\Gamma}-\Delta_T$ for the fixed class mean gap $\bar{\Gamma}=\mathbb{E}_H\theta-\mathbb{E}_L\theta$ (Lemma A.5). Let $J(r)\geq k$ and $r'>r$. Then $\Delta_T(r)[g_H(r)-c_H]=\gamma(r)J(r)>0$, and the balance condition gives

$$J(r')=\frac{\Delta_T(r')[g_H(r')-c_H]}{\gamma(r')}\geq\frac{\Delta_T(r)[g_H(r)-c_H]}{\gamma(r')}\geq\frac{\Delta_T(r)[g_H(r)-c_H]}{\gamma(r)}=J(r)\geq k.$$

So $\{J\geq k\}$ is an up-set on which J is nondecreasing. If $\Delta_T(r')>\Delta_T(r)$, then $\gamma(r')<\gamma(r)$ and the

second inequality is strict. By part (a), and because $\tau \le M < 1$ at every strength with coexistence, the strengths with coexistence are $\{\pi < \tau\} \cap \{J \ge k\} \cap \{\tau \le M\} \cap \{\Delta_T \le \Delta_T^{\mathrm{N}}\}$: two up-sets and two down-sets, by the proof of part (b). They form an interval, and the argument holds at every prior and every slot.

For (c)(ii), $\tau > \pi$ at the weakest strength and $\{\pi < \tau\}$ is an up-set, so it is all of I. Continuity of Δ_T and τ makes $\{J \ge k\}$, $\{\tau \le M\}$ and $\{\Delta_T \le \Delta_T^{\mathrm{N}}\}$ closed subsets of the compact interval I. A closed up-set of I is empty or runs from its least element to $\max I$, and a closed down-set is empty or runs from $\min I$ to its largest element. The set with coexistence is therefore empty or the closed interval from the weakest strength with $J \ge k$ to the smaller of the strongest strength with $\tau \le M$ and the strongest strength with $\Delta_T \le \Delta_T^{\mathrm{N}}$. □

Proof of Proposition 3(c)(iii). The distributions $U[0, r]$, $r \in I$, are continuous and ordered by first-order stochastic dominance. Write $\bar{\Gamma} = \mathbb{E}_H\theta - \mathbb{E}_L\theta$ for the class mean gap. Every $r \in I$ has $r \ge \min I > p$, so $(r^2 - p^2)/(2r^2) > 0$ on I. The proof has four steps.

Step 1: the auction terms. Fix $r \in I$ and let R be uniform on $[0, r]$. A high-class value $\theta \ge \max I \ge r$ wins against every incumbent value and pays $\max\{p, R\}$, whose mean is $r/2 + p^2/(2r)$. A low-class value $\theta \in [p, \min I] \subseteq [p, r]$ earns $\mathbb{E}(\theta - \max\{p, R\})_+ = (\theta^2 - p^2)/(2r)$, and the target receives θ minus this rent. Averaging over each class gives

$$g_H = \mathbb{E}_H\theta - \frac{r}{2} - \frac{p^2}{2r}, \qquad g_L = \frac{\mathbb{E}_L\theta^2 - p^2}{2r}, \qquad \Delta_T = \frac{r}{2} + \frac{p^2}{2r} - \mathbb{E}_L\theta + \frac{\mathbb{E}_L\theta^2 - p^2}{2r} = \frac{\mathbb{E}_L(r - \theta)^2}{2r},$$

so $\gamma = g_H - g_L = \mathbb{E}_H\theta - r/2 - \mathbb{E}_L\theta^2/(2r)$ and $\Delta_T + \gamma = \bar{\Gamma}$ on I. These functions are smooth in $r > 0$, and

$$\Delta_T' = \frac{r^2 - \mathbb{E}_L\theta^2}{2r^2} \ge 0, \qquad g_H' = -\frac{r^2 - p^2}{2r^2}, \qquad \gamma' = -\Delta_T',$$

where $\Delta_T' \ge 0$ because $\mathbb{E}_L\theta^2 \le (\min I)^2 \le r^2$. Also $\Delta_T(r) > 0$ for every $r > \min I$, because every low-class value is at most $\min I$.

Step 2: three inequalities. First, $\mathbb{E}_H\theta - c_H \ge \max I \ge r$, so

$$g_H - c_H = \mathbb{E}_H\theta - c_H - \frac{r}{2} - \frac{p^2}{2r} \ge \frac{r}{2} - \frac{p^2}{2r} = r\,\frac{r^2 - p^2}{2r^2} > 0.$$

Second, every low-class value lies in $[0, r]$, so

$$r\Delta_T' - \Delta_T = \mathbb{E}_L\theta - \frac{\mathbb{E}_L\theta^2}{r} = \mathbb{E}_L\left[\theta\left(1 - \frac{\theta}{r}\right)\right] \ge 0.$$

Third, $\mathbb{E}_H\theta \ge \max I + c_H \ge r + c_H$ and $\mathbb{E}_L\theta^2 \le r^2$, so $\gamma \ge (r + c_H) - r/2 - r^2/(2r) = c_H > 0$. Hence $g_H > g_L$ and $\bar{\Gamma} = \Delta_T + \gamma > 0$.

Step 3: the product. By Steps 1 and 2,

$$\big(\Delta_T\,(g_H - c_H)\big)' = \Delta_T'\,(g_H - c_H) - \frac{r^2 - p^2}{2r^2}\,\Delta_T \geq \frac{r^2 - p^2}{2r^2}\,(r\Delta_T' - \Delta_T) \geq 0,$$

so $\Delta_T(r)\,[g_H(r) - c_H]$ is nondecreasing on I.

Step 4: the return J. Since $J = \Delta_T\,(g_H - c_H)/\gamma$ and $\gamma' = -\Delta_T'$,

$$J' = \frac{\big(\Delta_T\,(g_H - c_H)\big)'\gamma + \Delta_T\,(g_H - c_H)\,\Delta_T'}{\gamma^2} = \frac{\bar{\Gamma}\,(g_H - c_H)\,\Delta_T' - \frac{r^2 - p^2}{2r^2}\,\Delta_T\,\gamma}{\gamma^2}$$
$$\geq \frac{r^2 - p^2}{2r^2} \cdot \frac{\bar{\Gamma}\,r\Delta_T' - \Delta_T\,\gamma}{\gamma^2} \geq \frac{r^2 - p^2}{2r^2} \cdot \frac{\Delta_T\,(\bar{\Gamma} - \gamma)}{\gamma^2} = \frac{(r^2 - p^2)\,\Delta_T^2}{2r^2\gamma^2}.$$

The first inequality uses $g_H - c_H \geq r\,(r^2 - p^2)/(2r^2)$ and $\Delta_T' \geq 0$; the second uses $r\Delta_T' \geq \Delta_T$ and $\bar{\Gamma} > 0$. The bound is nonnegative on I and positive wherever $\Delta_T > 0$, which includes every $r > \min I$ by Step 1, so J is strictly increasing on I. □

For point-mass classes at h and ℓ, the hypotheses of part (c)(iii) read $\ell \in [p, \min I]$, $p < \min I$ and $\max I + c_H \leq h$.

Proof of Proposition 3(c)(iv). Since $1 - \tau = (g_H - c_H)/\gamma$, $\Delta_T(1 - \tau)^2 = \Delta_T(g_H - c_H)^2/\gamma^2$. The numerator is nonnegative and nondecreasing by hypothesis, and $1/\gamma^2$ is positive and nondecreasing, because $\gamma > 0$ is nonincreasing; their product is nondecreasing.

By the pair (T) of Lemma A.7, when an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, the threshold pair needs $0 < s^* < \lambda$, that is $\pi < \tau < M$, and $\rho \leq k/J \leq 1$, that is $\rho J \leq k \leq J$. With entry ρ after a sale and after the zero order, its entry is $\rho + (k/J - \rho)(b + \pi s^*)$, and $b + \pi s^* = (1 - \pi)b/(1 - \tau)$, which gives the display. Let $r < r'$ be two strengths of this set. Both have $J \geq k$, so part (c)(i) gives $J(r') \geq J(r)$ and $0 \leq k - \rho J(r') \leq k - \rho J(r)$, and the first sentence gives $\Delta_T(r')[1 - \tau(r')]^2 \geq \Delta_T(r)[1 - \tau(r)]^2 > 0$. Hence $\mathsf{E}_\mathrm{T}(r') \leq \mathsf{E}_\mathrm{T}(r)$. Finally, $s^* = \lambda$ exactly when $\tau = M$, and the purchase at the rate λ is the pair (F). □

Proof of Proposition 3(d). Fix F and the slot, and let the prior be $1/2$. Since $\pi < \tau$, Lemmas A.7 to A.9 apply against F. The proof has five steps.

Step 1: the frame. At the prior $1/2$,

$$1 - \mu_+(s) = \frac{b}{2b + s} = \mu_-(s) \qquad \text{for every } s \in [0, \lambda],$$

and in particular $1 - M = m = b/(2b + \lambda)$. Each tuple below has no wrong-signed order, rates with $s_+ \geq s_-$, entry ρ after a sale and after the zero order, a stated entry $e(1)$ after a purchase, the prices

$P(x) = t_0 + e(x)\, w(\mu(x))$, and the rival's belief $\mu(x)$ at the price $P(x)$. By Lemma A.9, such a tuple is an equilibrium under a pair of tie rules when $(s_+, e(1))$ is a pair of Lemma A.7 and s_- is a rate of Lemma A.8 under those rules. Steps 2 to 5 establish these two conditions.

Step 2: the uninformative market. Let $\rho\Delta_T \leq 2k$, and take $s_+ = s_- = 0$ and entry ρ after every order. Every posterior equals $1/2 < \tau$. The pair $(0, \rho)$ is the pair (N) of Lemma A.7, because $\rho\Delta_T(1-\pi) = \rho\Delta_T/2 \leq k$, and the rate 0 is the rate (N) of Lemma A.8, because $\rho\Delta_T\pi = \rho\Delta_T/2 \leq k$. Neither condition involves a tie at the threshold, so the uninformative market is an equilibrium under either tie rule for the rival.

Step 3: the low-class rate. Let $\rho\Delta_T > 2k$. By Step 1, $\rho\Delta_T m \geq k$ holds exactly when $b\,(\rho\Delta_T/k - 2) \geq \lambda$. At the prior $1/2$ the rate $\hat{s}_-$ of Lemma A.8 is $b\,(\rho\Delta_T/k - 2)$ when $\rho\Delta_T m < k < \rho\Delta_T/2$, and λ when $\rho\Delta_T m \geq k$. Hence $\hat{s}_- = \min\{\lambda,\, b\,(\rho\Delta_T/k - 2)\} > 0$, the one rate of that lemma, and every tuple in Steps 4 and 5 has $s_- = \hat{s}_-$. Its gross return at entry ρ satisfies

$$\rho\Delta_T\, \mu_-(\hat{s}_-) \geq k, \tag{A.9}$$

with equality when $\hat{s}_- < \lambda$. Indeed, if $\hat{s}_- = b\,(\rho\Delta_T/k - 2)$, then $2b + \hat{s}_- = b\rho\Delta_T/k$ and $\mu_-(\hat{s}_-) = k/(\rho\Delta_T)$; if $\hat{s}_- = \lambda$, then $\rho\Delta_T\mu_-(\lambda) = \rho\Delta_T m \geq k$.

Step 4: a purchase at the rate $\hat{s}_-$ does not recruit. Let $\rho\Delta_T > 2k$ and $\mu_+(\hat{s}_-) < \tau$. This is the case whenever $\tau \geq 1$, because $\mu_+(\hat{s}_-) \leq M < 1$. Take $s_+ = \hat{s}_-$ and $e(1) = \rho$. If $\hat{s}_- < \lambda$, then $(\hat{s}_-, \rho)$ is the pair (Q) of Lemma A.7: $0 < \hat{s}_- < \lambda$, $\mu_+(\hat{s}_-) < \tau$, and $\rho\Delta_T[1 - \mu_+(\hat{s}_-)] = \rho\Delta_T\mu_-(\hat{s}_-) = k$ by Step 1 and (A.9). If $\hat{s}_- = \lambda$, then $M = \mu_+(\lambda) < \tau$, so (λ, ρ) is the pair (F), and its return condition $\rho\Delta_T(1 - M) = \rho\Delta_T m \geq k$ holds. The posteriors $\mu_-(\hat{s}_-) < \pi$, $\mu_0 = \pi$ and $\mu_+(\hat{s}_-)$ all lie below τ, so the tuple is an equilibrium under either tie rule for the rival, with entry ρ after every order.

Step 5: a purchase at the rate $\hat{s}_-$ recruits. Let $\rho\Delta_T > 2k$ and $\mu_+(\hat{s}_-) \geq \tau$, so that $\tau \leq \mu_+(\hat{s}_-) \leq M < 1$. Take $e(1) = 1$ and $s_+ = \min\{\lambda,\, b\,(\Delta_T/k - 2)\}$. By Step 1 and (A.9),

$$\Delta_T[1 - \mu_+(\hat{s}_-)] = \Delta_T\mu_-(\hat{s}_-) \geq k/\rho > k. \tag{A.10}$$

Since $\Delta_T(1-M) = \Delta_T b/(2b+\lambda)$, the inequality $\Delta_T(1-M) \geq k$ holds exactly when $b\,(\Delta_T/k-2) \geq \lambda$.

If $\Delta_T(1 - M) \geq k$, then $s_+ = \lambda \geq \hat{s}_-$, and $M = \mu_+(\lambda) \geq \mu_+(\hat{s}_-) \geq \tau$. When $M > \tau$, the pair (F) of Lemma A.7 has $e(1) = 1$; when $M = \tau$, either tie rule for the rival admits $e(1) = 1$. The return condition $e(1)\Delta_T(1 - M) = \Delta_T(1 - M) \geq k$ holds, so $(\lambda, 1)$ is the pair (F).

If $\Delta_T(1 - M) < k$, then $s_+ = b\,(\Delta_T/k - 2) < \lambda$, and $\hat{s}_- < \lambda$, because $\hat{s}_- = \lambda$ would give $\Delta_T(1 - M) = \Delta_T[1 - \mu_+(\hat{s}_-)] > k$ by (A.10). The return $\Delta_T[1 - \mu_+(s)] = \Delta_T b/(2b + s)$ is continuous and strictly decreasing in s. It exceeds k at $s = \hat{s}_-$ by (A.10), and it equals k exactly at $s = s_+$. Hence $\hat{s}_- < s_+ < \lambda$. Since μ_+ rises strictly, $\mu_+(s_+) > \mu_+(\hat{s}_-) \geq \tau$, so Lemma A.2(d) gives entry one after a purchase under either tie rule for the rival, and $(s_+, 1)$ is the pair (R) of Lemma A.7.

In both cases $s_+ \geq \hat{s}_- = s_-$. The posteriors after a sale and after the zero order are at most $1/2 < \tau$, so their entry is ρ under either tie rule for the rival, and Step 1 shows that the tuple is an equilibrium. □

Two intervals of coexistence. Without the balance condition, the strengths with coexistence along a path can form two disjoint intervals, even when class H outbids every incumbent and its value exceeds every incumbent value by more than c_H. Keep every primitive of the example except the incumbent distribution, take the prior $1/2$ and any slot with $141/200 < M(1/2) < 4245/5804$, the example's thresholds at r_1 and r_2, and for $r \in [0, 2]$ let F_r put weight 0.9 on a lower uniform distribution and the rest on an upper uniform distribution, both of half-width 0.01. The centre of the lower distribution rises linearly from 0.6 to 0.98 as r runs over $[0, 1]$ and then stays there; the centre of the upper distribution stays at 1.57 on $[0, 1]$ and rises linearly to 1.8 as r runs over $[1, 2]$. Every lower value lies between p and ℓ, and every upper value between ℓ and h. On $[0, 1]$ the incumbent strengthens only where it loses to both classes: both profits fall, the spread stays constant, τ rises and J falls. On $[1, 2]$ it strengthens between the class values, and the spread widens. An uninformative market and an informative market coexist exactly on $[0, r'] \cup [r'', 2]$, where $r' = 3137/6498 \approx 0.482764$ and $r'' = (1485 - \sqrt{2127361})/23 \approx 1.150106$ are the two strengths in $[0, 2]$ at which $J(r) = k$. The strengths with a sure purchase form one interval inside the second piece, and on the gap the spread lies below Δ_T^{F}.

Other tie rules. The results above take the default tie rules. The two alternatives change the region of part (a) as follows.

- When an indifferent high-cost rival may prepare with any probability, an uninformative market and an informative market coexist against F exactly when F satisfies (A.7), or when $\tau = \pi$ and $k/(1 - \pi) < \Delta_T \leq \Delta_T^{\mathrm{N}}$. At $\tau = \pi$, one uninformative market has entry ρ after every order, and one informative market is constructed as in part (a), with entry ρ after a sale. Inside the region, wherever $\tau < M$ and $J > k$, the threshold pair with $e(1) = k/J$ is a further informative market, and its entry lies strictly between ρ and (A.8). Where $\tau = M$, a full purchase is an equilibrium with each entry after a purchase in $[k/J, 1]$.

- When an indifferent informed trader abstains and an indifferent high-cost rival prepares, an uninformative market and an informative market coexist against F exactly when $\pi < \tau \leq M$ and $\Delta_T^{\mathrm{F}} < \Delta_T \leq \Delta_T^{\mathrm{N}}$, and the informative market is the full purchase. When an indifferent informed trader abstains and an indifferent high-cost rival may prepare with any probability, the distributions with $\tau = \pi$ and $\Delta_T^{\mathrm{F}} < \Delta_T \leq \Delta_T^{\mathrm{N}}$ join this set.

Proof of the statements on other tie rules. Each statement follows from Lemmas A.7 to A.9 and the steps of the proof of part (a).

An indifferent informed trader may mix, and an indifferent high-cost rival may prepare with any probability. If $\tau < \pi$, every posterior of an uninformative market lies strictly above τ, so Step 1 of the proof of part (a) applies with no tie, and coexistence fails. If $\pi < \tau$, Steps 2 and 3 hold as written: the pairs (N) and (Q) involve no tie at τ, the pairs that need $\mu_+(s_+) \geq \tau$ still need $\tau \leq M$ and $J \geq k$, and the informative pair constructed in Step 3 remains available, the full purchase with $e(1) = 1$ at $\tau = M$ and the threshold pair with $e^* = 1$ at $J = k$. The coexistence set for $\pi < \tau$ is therefore (A.7). Inside the region, where $\tau < M$ and $J > k$, the threshold pair has $0 < s^* < \lambda$ and $e^* = k/J < 1$; since $\tau > \pi$ and $\Delta_T \leq \Delta_T^{\mathrm{N}}$, $\rho J < \rho\Delta_T(1-\pi) \leq k$, so $e^* > \rho$. With $s_- = 0$, Lemma A.9 makes it an equilibrium. Its entry $\rho + (e^* - \rho)(b + \pi s^*)$ exceeds ρ, and it falls short of (A.8), which is $\rho + (1-\rho)(b + \pi s_+)$, because $e^* < 1$ and $s^* < s_+$, the return at s^* being $J > k$. Where $\tau = M$, $J = \Delta_T(1-M) \geq k$ and $\rho J < k$, and the full purchase meets its return condition exactly when $e(1)J \geq k$, that is for $e(1) \in [k/J, 1]$.

Let $\tau = \pi$. Every uninformative market has entry at least ρ, so it needs $\rho\Delta_T \max\{\pi, 1-\pi\} \leq k$, that is $\Delta_T \leq \Delta_T^{\mathrm{N}}$; conversely, entry ρ after every order is allowed at the posterior $\pi = \tau$, and it gives an uninformative market. Under $\Delta_T \leq \Delta_T^{\mathrm{N}}$, a sale at a positive rate has a posterior below τ, entry ρ and a gross return below $\rho\Delta_T\pi \leq k$, so $s_- = 0$ in every equilibrium. A purchase at a positive rate has a posterior above τ, so $e(1) = 1$, and its return $\Delta_T[1 - \mu_+(s_+)]$ falls strictly from $\Delta_T(1-\pi)$ at $s_+ = 0$ to $\Delta_T(1-M)$ at $s_+ = \lambda$. An informative market therefore exists exactly when $\Delta_T(1-\pi) > k$: a full purchase when $\Delta_T(1-M) \geq k$, and otherwise a mixed purchase at the root in $(0, \lambda)$. Its posteriors satisfy $\mu_0 < \pi = \mu_-(0) < \mu_+(s_+)$; with entry ρ after the zero order and, at the tie, after the sale, the sufficiency argument of Lemma A.9 applies.

An indifferent informed trader abstains. Then (Q), (R) and (T) drop out of Lemma A.7, the pair (F) needs a strict return, and the rate (N) of Lemma A.8 is unchanged. Step 1 of the proof of part (a) uses strict inequalities only, so it excludes $\tau < \pi$, and it excludes $\tau = \pi$ when an indifferent high-cost rival prepares. Let $\pi < \tau$ and $\Delta_T \leq \Delta_T^{\mathrm{N}}$. The low class abstains, because a full sale earns $\rho\Delta_T m < \rho\Delta_T\pi \leq k$, and the informative market is a full purchase with $\mu_+(\lambda) = M \geq \tau$, entry one after it, and $\Delta_T(1-M) > k$, that is $\Delta_T > \Delta_T^{\mathrm{F}}$; a rival who may prepare with any probability admits entry one at $M = \tau$ as well. When it may prepare with any probability and $\tau = \pi$, the argument for $\tau = \pi$ above applies with the full purchase as the only informative pair, so $\Delta_T^{\mathrm{F}} < \Delta_T \leq \Delta_T^{\mathrm{N}}$. □

The example

The path and its marked strengths. In the example the classes are point masses at h and ℓ, the incumbent is uniform on $[0, r]$, and $(h, \ell, p, \rho, c_H, k) = (10, 1, 1/2, 1/4, 6, 1/50)$. The path is $r \in [r_0, r_2]$, with $(r_0, r_1, r_2) = (6/5, 3, 18/5)$. On it $\ell < r < h$, so $\Delta_T(r) > 0$ and $g_H(r) > g_L(r)$.

On the path $\Delta_T(r) = (r-\ell)^2/(2r)$, with $\Delta_T'(r) = \frac{1}{2} - \ell^2/(2r^2) > 0$, and the gross return of a purchase that just reaches $\tau(r)$ and recruits is

$$J(r) = \Delta_T(r)\,[1-\tau(r)] = \frac{(r-\ell)^2\,[2r(h-c_H) - r^2 - p^2]}{2r\,(2rh - r^2 - \ell^2)}.$$

Write $\mathfrak{r}(y) = \ell + y + \sqrt{y^2 + 2\ell y}$, the root above ℓ of $\Delta_T(r) = y$; $\Delta_T(r) \le y$ holds exactly when $r \le \mathfrak{r}(y)$. At the prior $1/2$ write $M = M(1/2)$ and $m = 1 - M$. Four marked strengths do not depend on the slot: r_A, the root of $J(r) = k$ in (r_0, r_2); $r_N = \mathfrak{r}(2k/\rho)$; r_J, the root of $\rho J(r) = k$ in (r_0, r_2); and r_*, the root of $\rho\Delta_T(r)\tau(r) = k$ in (r_0, r_2). For a slot with $0 < \lambda < 1$, $0 < \beta < 1/2$ and $\tau(r_1) < M < \tau(r_2)$, three more are $r_F = \mathfrak{r}\big(k/(1-M)\big)$, $r_S = \mathfrak{r}\big(k/(\rho m)\big)$, and r_C, the root of $\tau(r) = M$ in (r_1, r_2). Lemma A.10 makes r_A, r_J, r_C and r_* unique.

At r_A a purchase that just reaches τ and recruits earns k; at r_J the same purchase without recruitment earns k; at r_N a unit order against an uninformative market earns $\rho\Delta_T/2 = k$; at r_F a full purchase that recruits earns $\Delta_T(1-M) = k$; and at r_S a full sale earns $\rho\Delta_T m = k$. The root r_C of $\tau(r) = M$, that is of $B_r(M) = c_H$, has the closed form

$$r_C = \frac{Mh - c_H + \sqrt{(Mh - c_H)^2 + M[(1-M)\ell^2 - p^2]}}{M}.$$

Lemma A.10 (The example path)**.** *On the path $[r_0, r_2]$ of the example:*

(a) τ is strictly increasing, with $\tau(r_0) = 195/308 > 1/2$ and $\tau(r_2) = 4245/5804$, so that $1 - \rho - \tau(r_2) = 27/1451 > 0$.

(b) The path is a uniform incumbent of Proposition 3(c)(iii), with $h - r_2 - c_H = 2/5 > 0$. The product $\Delta_T(r)\,[g_H(r) - c_H]$ has derivative at least $9667/8280 \cdot \Delta_T'(r) > 0$ at every r. Hence J is strictly increasing, with $J(r_0) = 113/18480 < k$ and $\rho J(r_2) = 263471/4178880 > k$.

(c) The product $\Delta_T(r)\,[g_H(r) - c_H]^2$ has derivative at least $937/5520 \cdot [g_H(r) - c_H]\,\Delta_T'(r) > 0$ at every r, and $\Delta_T(r)\,[1-\tau(r)]^2$ is strictly increasing on the whole path.

(d) $\Delta_T(r)\,\tau(r)$ is strictly increasing.

Proof. On the path $\ell < r < h$, and Step 1 of the proof of Proposition 3(c)(iii) gives

$$g_H = h - \frac{r}{2} - \frac{p^2}{2r}, \qquad g_L = \frac{\ell^2 - p^2}{2r}, \qquad \Delta_T = \frac{(r-\ell)^2}{2r}, \qquad \gamma = \frac{2rh - r^2 - \ell^2}{2r}.$$

(a) The threshold $\tau = (c_H - g_L)/\gamma$ is the ratio of the numerator $2rc_H - \ell^2 + p^2$ to the denominator $2rh - r^2 - \ell^2$. The denominator exceeds $r^2 - \ell^2 > 0$, because $2h - r > r$. The derivative of τ has

the sign of the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, which is

$$2c_H(r^2 - \ell^2) + 2(\ell^2 - p^2)(h - r) > 0,$$

since $\ell < r < h$ and $\ell \geq p$. The two values follow from the closed form of τ, and the last from $\rho = 1/4$.

(b) The interval $I = [r_0, r_2]$ has $p < \ell < r_0$ and $h > r_2$, and the margin $h - r_2 - c_H$ is the stated one, so the path is a uniform incumbent of Proposition 3(c)(iii). Directly, $g'_H = -(r^2 - p^2)/(2r^2)$ and $\Delta_T/\Delta'_T = r(r - \ell)/(r + \ell)$, so

$$\big(\Delta_T\,(g_H - c_H)\big)' = \Delta'_T(g_H - c_H) + \Delta_T g'_H = \Delta'_T\Big[g_H - c_H - \frac{(r^2 - p^2)(r - \ell)}{2r(r + \ell)}\Big].$$

In the bracket, $g_H - c_H$ falls strictly, because $g'_H < 0$ for $r > p$, and the subtracted term rises strictly, as the product of the positive increasing functions $(r^2 - p^2)/(2r)$ and $(r - \ell)/(r + \ell)$. The bracket is therefore smallest at r_2, where it equals $9667/8280$, and this gives the bound. In particular $g_H - c_H > 0$ on the path, so the numerator $\Delta_T(g_H - c_H)$ of $J = \Delta_T(g_H - c_H)/\gamma$ is positive and strictly increasing, and γ is positive and strictly decreasing, since $\gamma' = -\Delta'_T < 0$. Hence J is strictly increasing. The two values follow from the closed form of J.

(c) In the same way,

$$\big(\Delta_T\,(g_H - c_H)^2\big)' = (g_H - c_H)\,\Delta'_T\Big[g_H - c_H - \frac{(r^2 - p^2)(r - \ell)}{r(r + \ell)}\Big],$$

and the bracket, strictly decreasing for the same reasons, equals $937/5520$ at r_2. Since $g_H - c_H > 0$, this gives the bound, which is positive. Division by γ^2, positive and strictly decreasing, shows that $\Delta_T(1 - \tau)^2 = \Delta_T(g_H - c_H)^2/\gamma^2$ is strictly increasing on the whole path.

(d) Δ_T and τ are positive and strictly increasing on the path, by $\Delta'_T > 0$ and (a), so their product is strictly increasing. □

By part (b), $J(r_0) < k < \rho J(r_2) < J(r_2)$ and $\rho J(r_0) < k$, so the roots r_A of $J = k$ and r_J of $\rho J = k$ exist in (r_0, r_2), and they are unique because J is strictly increasing. The root r_C is unique because τ is strictly increasing, and r_* because $\Delta_T\tau$ is.

Lemma A.11 (The order of the marked strengths)**.** *For every slot with* $0 < \lambda < 1$, $0 < \beta < 1/2$ *and* $\tau(r_1) < M(1/2) < \tau(r_2)$,

$$r_0 < r_A < r_F < r_N < r_J < r_S < r_1 < r_C < r_2.$$

Proof. Each inequality reduces to a fact of Lemma A.10 or to the band $\tau(r_1) < M < \tau(r_2)$. Every strength in the chain exceeds ℓ, where Δ_T is strictly increasing.

- $r_0 < r_A < r_1$: $J(r_0) < k < \rho J(r_1) < J(r_1)$, with $\rho J(r_1) = 59/1200$, and J is increasing.
- $r_A < r_F$: $\Delta_T(r_A) < k/(1-M)$ is $(1-M)\Delta_T(r_A) < k = (1-\tau(r_A))\Delta_T(r_A)$, that is $M > \tau(r_A)$. This holds because $M > \tau(r_1) > \tau(r_A)$, since $r_A < r_1$.
- $r_F < r_N$: $k/(1-M) < 2k/\rho$ is $M < 1 - \rho/2$. This holds because $M < \tau(r_2) < 1 - \rho$, by Lemma A.10(a).
- $r_N < r_J$: $\rho\Delta_T(r_J)[1-\tau(r_J)] = k$ and $\tau(r_J) > \tau(r_0) > 1/2$ give $\rho\Delta_T(r_J) > 2k$, that is $\Delta_T(r_J) > \Delta_T(r_N)$.
- $r_J < r_S$: $\Delta_T(r_J) < k/(\rho m)$ is $\rho(1-M)\Delta_T(r_J) < k = \rho(1-\tau(r_J))\Delta_T(r_J)$, that is $M > \tau(r_J)$. The value $\rho J(r_1) > k$ gives $r_J < r_1$, so $M > \tau(r_1) > \tau(r_J)$.
- $r_S < r_1$: the claim is $\rho m \Delta_T(r_1) > k$. By Lemma A.10(a), $m = 1 - M > 1 - \tau(r_2) > \rho$, so $\rho m \Delta_T(r_1) > \rho^2 \Delta_T(r_1)$, and $\rho^2\Delta_T(r_1) - k = 13/600 > 0$.
- $r_1 < r_C < r_2$: this is the band, since τ is increasing. □

The root r_* lies in (r_A, r_N). Write $\tau_A = \tau(r_A)$. Since $J(r_A) = k$, $\Delta_T(r_A) = k/(1-\tau_A)$, and $\rho\Delta_T(r_A)\tau_A < k$ is $\rho\tau_A < 1 - \tau_A$, which holds because $(1+\rho)\tau_A < (1+\rho)(1-\rho) < 1$ by Lemma A.10(a). At r_N, $\rho\Delta_T(r_N)\tau(r_N) = 2k\,\tau(r_N) > k$.

Table A.1 gives the marked strengths at the example's slot, $\beta = 0.294655$ and $1 + u = \mathrm{e}$, so that $\lambda = 0.336122$. The roots r_A, r_J and r_* are unique by Lemma A.10; r_F, r_N, r_S and r_C have the closed forms above. Each value is rounded to six places.

Table A.1. Marked strengths of the example at the prior $1/2$.

Strength	Definition	Value in the example	Depends on the slot
r_A	$J(r) = k$	1.393958	no
r_F	$\Delta_T(r)(1-M) = k$	1.467127	yes
r_N	$\rho\Delta_T(r)/2 = k$	1.747878	no
r_J	$\rho J(r) = k$	1.966097	no
r_S	$\rho\Delta_T(r)m = k$	2.124148	yes
r_C	$\tau(r) = M$	3.592659	yes
r_*	$\rho\Delta_T(r)\tau(r) = k$	1.634218	no

The forms at the prior $1/2$**.** Take the prior $1/2$ and a slot with $\tau(r_1) < M < \tau(r_2)$. Then $1 - \mu_+(s_+) = b/(2b + s_+)$, $\mu_-(s_-) = b/(2b + s_-)$ and $s^* = b(2\tau - 1)/(1 - \tau)$. Under the default tie rules the low class sells at the rate of Lemma A.8,

$$\hat{s}_-(r) = 0 \text{ on } [r_0, r_N], \qquad \hat{s}_-(r) = b\,[\rho\Delta_T(r)/k - 2] \text{ on } (r_N, r_S), \qquad \hat{s}_-(r) = \lambda \text{ on } [r_S, r_2],$$

and class H's pair takes one of five forms:

- **N**, an uninformative market: $s_+ = 0$ and $e(1) = \rho$;
- **R**, a mixed purchase that recruits: $s_+ = s_R(r) := b[\Delta_T(r)/k - 2]$ and $e(1) = 1$;
- **F**, a full purchase: $s_+ = \lambda$, with $e(1) = 1$ for $r < r_C$, $e(1) = 1$ at r_C because an indifferent high-cost rival prepares, and $e(1) = \rho$ for $r > r_C$;
- **Q**, mixed orders at the entry floor: $s_+ = s_Q(r) := b[\rho\Delta_T(r)/k - 2] = \hat{s}_-(r)$ and $e(1) = \rho$;
- **T**, the threshold pair: $s_+ = s^*(r)$ and $e(1) = e^*(r) = k/J(r)$, which requires $J(r) = k$, that is $r = r_A$.

The forms R, Q and T need an indifferent informed trader who may mix. When an indifferent high-cost rival may prepare with any probability, a full purchase at r_C admits every $e(1) \in [\rho, 1]$, and the threshold pair is a pair of Lemma A.7 exactly on $[r_A, r_J]$, where $\rho J(r) \le k \le J(r)$.

Each form defines the equilibrium tuple completely:

- orders: $\sigma_H(1) = s_+/\lambda$, $\sigma_H(0) = 1 - s_+/\lambda$, $\sigma_L(-1) = \hat{s}_-(r)/\lambda$, $\sigma_L(0) = 1 - \hat{s}_-(r)/\lambda$;
- posteriors: $\mu_-(\hat{s}_-)$, $\mu_0(s_+, \hat{s}_-)$ and $\mu_+(s_+)$ after the orders -1, 0 and 1;
- beliefs of the rival: $\bar{\mu}(P(x)) = \mu(x)$ at each price;
- high-cost preparation: $\zeta(P(1)) = (e(1) - \rho)/(1 - \rho)$, and $\zeta(P(x)) = 0$ at the prices of the orders 0 and -1;
- prices: $P(x) = t_0 + e(x)\, w(\mu(x))$;
- bids: truthful.

If two orders share a price, they share the posterior and the entry (Lemma A.9), so ζ is well defined as a function of the price.

Corollary 2 (Coexistence in the example, full statement). *Take the example at the prior* $1/2$*, a slot with* $0 < \lambda < 1$*,* $0 < \beta < 1/2$ *and* $\tau(r_1) < M(1/2) < \tau(r_2)$*, and the default tie rules. The equilibrium set at each strength of the path is*

Strength	*Equilibria*
$r_0 \le r < r_A$	**N**
$r = r_A$	**N**; **T** *with entry one after a purchase*
$r_A < r < r_F$	**N**; **R**
$r_F \le r \le r_N$	**N**; **F**
$r_N < r < r_J$	**F**; **Q**
$r_J \le r \le r_2$	**F**

In each, the low class sells at the rate $\hat{s}_-(r)$. *Entry is* ρ *in an uninformative market and in mixed orders at the entry floor;* $\mathsf{E}_{\mathrm{R}}(r) = \rho + (1-\rho)\, b\Delta_T(r)/(2k)$ *in a mixed purchase that recruits, and* $\mathsf{E}_{\mathrm{R}}(r_A)$ *in the threshold pair at* r_A*; and* $\mathsf{E}_{\mathrm{F}} = \rho + (1-\rho)(b + \lambda/2)$ *in a full purchase up to* r_C*, with* $\mathsf{E}_{\mathrm{F}} = \rho$ *above* r_C.

(a) An uninformative market and an informative market coexist exactly on $[r_A, r_N]$.

(b) Along the informative branch, entry is $\rho + (1-\rho)\min\{b\Delta_T(r)/(2k),\ b + \lambda/2\}$ *on* $[r_A, r_C]$ *and* ρ *on* $(r_C, r_2]$. *It is continuous on* $[r_A, r_C]$*, rises strictly on* $[r_A, r_F]$*, is constant on* $[r_F, r_C]$*, and returns to the floor* ρ *above the entry ceiling* r_C.

(c) The equilibrium set is nonempty at every strength.

Proof of Corollary 2. Fix a slot in the band and the prior $1/2$. The proof has eight steps. Steps 2 and 3 classify the pairs and the rates under each pair of tie rules; Step 5 assembles the set under the default tie rules.

Step 1: the frame. By Lemma A.10(a), $\tau(r) \ge \tau(r_0) > 1/2 = \pi$ on the path, so Lemmas A.7 to A.9 apply at every r. Every equilibrium has a high-class pair from Lemma A.7 and a low-class rate from Lemma A.8. Step 4 shows $s_+ \ge s_-$ for every such combination, and Lemma A.9 then gives the equilibrium set as their product. Recall $1 - M = 1/(2+u)$ and $ub = \lambda$.

Step 2: the high-class pairs. Use Lemma A.10, the definitions of the marked strengths, and Lemma A.11.

- (N) holds if and only if $\rho\Delta_T/2 \le k$, that is $r \le r_N$.
- (R): $\Delta_T b/(2b + s_+) = k$ gives $s_+ = s_R(r)$. Then $\mu_+(s_R) > \tau$ if and only if $k/\Delta_T < 1 - \tau$, that is $J(r) > k$, that is $r > r_A$. Also $s_R < \lambda$ if and only if $\Delta_T/k < 2 + u$, that is $\Delta_T(1 - M) < k$, that is $r < r_F$. So (R) holds if and only if $r_A < r < r_F$, with an indifferent informed trader who may mix.
- (Q): $\rho\Delta_T b/(2b + s_+) = k$ gives $s_+ = s_Q(r)$. Then $s_Q > 0$ if and only if $r > r_N$; $\mu_+(s_Q) < \tau$ if and only if $\rho J(r) < k$, that is $r < r_J$; and $s_Q < \lambda$ if and only if $\rho\Delta_T(1 - M) < k$, that is $r < r_S$.

Since $r_J < r_S$, (Q) holds if and only if $r_N < r < r_J$, with an indifferent informed trader who may mix.

- (T): $s^* \in (0, \lambda)$ if and only if $1/2 < \tau(r) < M$, that is $r < r_C$. Also $\rho \le e^* \le 1$ if and only if $\rho J(r) \le k \le J(r)$, that is $r_A \le r \le r_J$, and $e^* = 1$ if and only if $r = r_A$. Since $r_J < r_C$, (T) holds at r_A when an indifferent high-cost rival prepares, and on $[r_A, r_J]$ when it may prepare with any probability, with an indifferent informed trader who may mix in both cases.

- (F): for $r < r_C$, $M > \tau(r)$, so $e(1) = 1$, and the condition is $\Delta_T(1 - M) \ge k$, that is $r \ge r_F$ ($r > r_F$ when an indifferent informed trader abstains). For $r > r_C$, $M < \tau(r)$, so $e(1) = \rho$, and $\rho\Delta_T(1 - M) > k$ holds because $r > r_C > r_S$. At $r = r_C$, an indifferent high-cost rival who prepares gives $e(1) = 1$; one who may prepare with any probability allows any $e(1) \in [\rho, 1]$, and every such value meets the condition because $\rho\Delta_T(r_C)(1 - M) > k$. So (F) holds if and only if $r \ge r_F$ when an indifferent informed trader may mix, and $r > r_F$ when it abstains.

Step 3: the low-class rate. At the prior $1/2$, Lemma A.8 gives, when an indifferent informed trader may mix, $s_- = 0$ if and only if $r \le r_N$, $s_- = \lambda$ if and only if $r \ge r_S$, and $\hat{s}_-(r) = b(\rho\Delta_T/k - 2)$ on (r_N, r_S). When an indifferent informed trader abstains, no rate qualifies if and only if $\rho\Delta_T m \le k < \rho\Delta_T/2$, that is on $(r_N, r_S]$.

Step 4: $s_+ \ge s_-$. For $r \le r_N$, $s_- = 0 \le s_+$. For $r > r_N$, the pairs are (F) with $s_+ = \lambda \ge s_-$; (Q) with $s_+ = s_Q(r) = \hat{s}_-(r)$; and (T) with $r_N < r \le r_J$. For (T), $s^* \ge \hat{s}_-(r)$ if and only if $(2\tau - 1)/(1 - \tau) \ge \rho\Delta_T/k - 2$, that is $1/(1 - \tau) \ge \rho\Delta_T/k$, that is $\rho J(r) \le k$, which holds for $r \le r_J$. By Lemma A.9, entry after the zero order and after a sale is ρ in every equilibrium.

Step 5: the set. Combine Steps 2 to 4 with the order of Lemma A.11. Under the default tie rules the set is: an uninformative market on $[r_0, r_A)$; an uninformative market and the threshold pair with $e^* = 1$ at r_A; an uninformative market and a mixed purchase that recruits on (r_A, r_F); an uninformative market and a full purchase on $[r_F, r_N]$; a full purchase and mixed orders at the entry floor on (r_N, r_J); and a full purchase on $[r_J, r_2]$. In each, the low class sells at the rate $\hat{s}_-(r)$ of Step 3. On $[r_0, r_N]$ this is Proposition 3(a) on the path, since there $\Delta_T(r) \le \Delta_T^{\mathrm{N}}$, $\pi < \tau(r) < M$, and $J(r) \ge k$ exactly when $r \ge r_A$.

Step 6: the tuples. Fix a form and a strength at which Steps 2 and 3 admit it. Step 4 gives $s_+ \ge s_-$, so the sufficiency part of Lemma A.9 shows that the tuple satisfies (E1) to (E6). In particular, if two orders have equal posteriors, one of them is 0 or -1, so the common posterior is at most $1/2 < \tau$, and both orders have entry ρ and the same price.

Step 7: entry. Entry is $\mathsf{E} = \sum_x \Pr(X = x)\, e(x)$. Since $e(0) = e(-1) = \rho$ and $\Pr(X = 1) = \frac{1}{2}(b + s_+) + \frac{1}{2}b = b + s_+/2$, $\mathsf{E} = \rho + [e(1) - \rho](b + s_+/2)$.

- A mixed purchase that recruits: $b + s_R/2 = b\Delta_T/(2k)$ and $e(1) = 1$, so $\mathsf{E}_{\mathrm{R}}(r) = \rho + (1 - \rho)\, b\Delta_T(r)/(2k)$. It rises strictly in r because Δ_T does. At r_F, $\Delta_T(r_F) = k/(1 - M) = k(2 + u)$,

so $b\Delta_T(r_F)/(2k) = b + ub/2 = b + \lambda/2$, and the mixed purchase joins the full purchase continuously.

- A full purchase: $b + \lambda/2$, with $e(1) = 1$ up to r_C and $e(1) = \rho$ above r_C. So $\mathsf{E}_\mathrm{F} = \rho + (1-\rho)(b + \lambda/2)$ on $[r_F, r_C]$ and $\mathsf{E}_\mathrm{F} = \rho$ on $(r_C, r_2]$.
- An uninformative market and mixed orders at the entry floor: $e(1) = \rho$, so $\mathsf{E} = \rho$.
- The threshold pair at r_A: $J(r_A) = k$ gives $\Delta_T(r_A)/k = 1/(1-\tau(r_A))$, so $s^*(r_A) = s_R(r_A)$, and with $e^* = 1$ its entry is $\mathsf{E}_\mathrm{R}(r_A)$.

Along the informative branch, entry is therefore $\rho + (1-\rho)\min\{b\Delta_T(r)/(2k),\ b + \lambda/2\}$ on $[r_A, r_C]$: it is continuous, rises strictly on $[r_A, r_F]$ because Δ_T does, and is constant on $[r_F, r_C]$. Above r_C it is ρ. The strict rise uses a mixed purchase, which needs an indifferent informed trader who may mix.

Step 8: existence. Proposition 3(d) applies at every strength, since $\tau(r) > 1/2$ by Step 1. Directly, an uninformative market exists on $[r_0, r_N]$ and a full purchase on $[r_F, r_2]$, and $r_F < r_N$.

Parts (a), (b) and (c) are Steps 5, 7 and 8. □

The example's slot. Table A.2 gives the equilibrium set at the example's slot under the default tie rules. In a mixed purchase that recruits, entry is 0.459588 at $r = 1.40$ and 0.506112 at $r = 1.45$. The low class abstains on $[r_0, r_N]$, mixes on (r_N, r_S) and sells for sure on $[r_S, r_2]$. The strengths 1.55, 1.60 and 1.65 lie in $[r_F, r_N]$, where the equilibria are an uninformative market and a full purchase under the default tie rules. When an indifferent high-cost rival may prepare with any probability, the threshold pair adds entry 0.341462, 0.321311 and 0.305208 at these three strengths. The first two values are E_R of Corollary 2, and the last three are E_T of Proposition 3(c)(iv) at the prior $1/2$, each at the stated strength.

Table A.2. The equilibrium set of the example at the prior $1/2$ and the example's slot, under the default tie rules.

Strengths	Equilibria	Entry
$[r_0,\ 1.393958)$	**N**	0.250000
1.393958	**N**; **T** with $e^* = 1$	0.250000; 0.454185
(1.393958, 1.467127)	**N**; **R**	0.250000; from 0.454185 up to 0.522757
[1.467127, 1.747878]	**N**; **F**	0.250000; 0.522757
(1.747878, 1.966097)	**F**; **Q**	0.522757; 0.250000
[1.966097, 3.592659]	**F**	0.522757
$(3.592659,\ r_2]$	**F**	0.250000

Coexistence at other priors. For a prior π, write $r_N(\pi) = \mathfrak{r}\big(k/(\rho \max\{\pi, 1-\pi\})\big)$. When some strength on the path has $\tau(r) \le M(\pi)$, let $r_C(\pi)$ be the largest one. When $\tau(r_0) \le \pi \le \tau(r_2)$, let r_π solve $\tau(r_\pi) = \pi$. Write $\tau_A = \tau(r_A)$. Three priors divide the cases: π_A^{lo} solves $M(\pi_A^{\mathrm{lo}}) = \tau_A$, that is $\pi_A^{\mathrm{lo}} = \tau_A/[\tau_A + (1-\tau_A)(1+u)]$; $\pi_A^{\mathrm{hi}} = \tau_A$; and $\pi_{\max} = \tau(r_*)$.

Corollary A.2 (Coexistence at every prior)**.** *Take the example, any slot with $0 < \lambda < 1$ and $0 < \beta < 1/2$, any prior $\pi \in (0,1)$, and the default tie rules. The set $C(\pi)$ of strengths on the path at which an uninformative market and an informative market coexist is the set of Proposition 3(a) on the path,*

$$C(\pi) = \big\{ r \in [r_0, r_2] : \; r \ge r_A, \;\; \pi < \tau(r) \le M(\pi), \;\; \rho \Delta_T(r) \max\{\pi, 1-\pi\} \le k \big\}.$$

(a) If $\pi < \pi_A^{\mathrm{lo}}$, $C(\pi)$ is empty. At $\pi = \pi_A^{\mathrm{lo}}$, $C(\pi) = \{r_A\}$.

(b) If $\pi_A^{\mathrm{lo}} < \pi < \pi_A^{\mathrm{hi}}$, then $C(\pi) = [r_A, \min\{r_N(\pi), r_C(\pi)\}]$, a nondegenerate interval. Coexistence starts at r_A for every prior in this window and every slot.

(c) If $\pi_A^{\mathrm{hi}} \le \pi < \pi_{\max}$, then $C(\pi) = (r_\pi, \min\{r_N(\pi), r_C(\pi)\}]$, a nonempty interval with $r_\pi \ge r_A$. Coexistence starts where the prior reaches the threshold.

(d) If $\pi \ge \pi_{\max}$, $C(\pi)$ is empty.

At each $r \in C(\pi)$ the informative market is a mixed purchase that recruits or a full purchase, and the threshold pair at r_A in case (b). Its entry is $\rho + (1-\rho)\min\{(1-\pi)\, b\Delta_T(r)/k, \; b + \pi\lambda\}$, which rises strictly while the purchase is mixed and is then constant.

Proof of Corollary A.2. Fix a slot and a prior. Every distribution of the path satisfies the hypotheses of Proposition 3, since $\ell < r < h$. By Proposition 3(a), coexistence holds at r if and only if $\pi < \tau(r) \le M(\pi)$ and $k/(1-\tau(r)) \le \Delta_T(r) \le \Delta_T^{\mathrm{N}}$. Under $\tau(r) \le M(\pi) < 1$, the first spread inequality is $J(r) \ge k$, which holds exactly when $r \ge r_A$ by Lemma A.10(b), and the second is $\rho\Delta_T(r)\max\{\pi, 1-\pi\} \le k$. This gives the display. By the same part, the informative market is a full purchase where $\Delta_T(r)[1 - M(\pi)] \ge k$, the threshold pair with entry one at r_A where $\Delta_T(r_A)[1-M(\pi)] < k$, and a mixed purchase that recruits elsewhere; its entry is (A.8) at $\Delta_T = \Delta_T(r)$, which rises strictly while the purchase is mixed, because Δ_T rises strictly on the path, and is then constant.

By Lemma A.10(a), $\pi < \tau(r)$ holds for every r if $\pi < \tau(r_0)$, for $r > r_\pi$ if $\tau(r_0) \le \pi \le \tau(r_2)$, and for no r if $\pi > \tau(r_2)$. The condition $\tau(r) \le M(\pi)$ holds for $r \le r_C(\pi)$ when $r_C(\pi)$ exists, and the last condition holds for $r \le r_N(\pi)$. Also $\rho\Delta_T(r_A)\max\{\pi, 1-\pi\} < \rho\Delta_T(r_A) = \rho k/(1-\tau_A) < k$, because $\tau_A < \tau(r_2) < 1 - \rho$ by Lemma A.10(a). So $r_A < r_N(\pi)$ for every prior.

(a) M is increasing in π, so $\pi < \pi_A^{\mathrm{lo}}$ gives $M(\pi) < \tau_A \le \tau(r)$ for every $r \ge r_A$, and no strength qualifies. At $\pi = \pi_A^{\mathrm{lo}} < \tau_A$ only r_A qualifies, and there $M(\pi) = \tau_A$: the informative market is the full purchase, with $\Delta_T(r_A)(1 - \tau_A) = k$.

(b) Let $\pi_A^{\mathrm{lo}} < \pi < \pi_A^{\mathrm{hi}}$; the window is nonempty because $u > 0$. For $r \ge r_A$, $\tau(r) \ge \tau_A > \pi$. Also $\tau_A < M(\pi)$ gives $r_A < r_C(\pi)$, and $r_A < r_N(\pi)$. So $C(\pi) = [r_A, \min\{r_N(\pi), r_C(\pi)\}]$, a nondegenerate interval. At r_A, $\Delta_T(r_A)[1 - M(\pi)] < \Delta_T(r_A)(1 - \tau_A) = k$, so the informative market is the threshold pair with entry one. The start r_A does not depend on the prior in this window or on the slot.

(c) Let $\pi_A^{\mathrm{hi}} \le \pi < \pi_{\max}$. The window is nonempty, because $r_A < r_*$ and τ rises strictly. Here $\tau(r_0) < \tau_A \le \pi < \tau(r_*) < \tau(r_2)$, so r_π exists, $r_\pi \ge r_A$, and $\pi < \tau(r)$ holds exactly for $r > r_\pi$. The interval $C(\pi) = (r_\pi, \min\{r_N(\pi), r_C(\pi)\}]$ is nonempty: $r_\pi < r_C(\pi)$ because $\tau(r_\pi) = \pi < M(\pi)$; and since $\pi > 1/2$, $r_\pi < r_N(\pi)$ if and only if $\rho\Delta_T(r_\pi)\tau(r_\pi) < k$, that is $r_\pi < r_*$ by Lemma A.10(d), that is $\pi < \pi_{\max}$.

(d) Let $\pi \ge \pi_{\max}$. Then $\pi > 1/2$, and any r with $\pi < \tau(r)$ has $r > r_\pi \ge r_*$, so $\rho\Delta_T(r)\pi > \rho\Delta_T(r_*)\tau(r_*) = k$, and no strength qualifies. □

The priors π_A^{hi} and $\pi_{\max}$ do not depend on the slot. In the example $\pi_A^{\mathrm{hi}} \approx 0.640739$ and $\pi_{\max} \approx 0.650059$, and at the example's slot $\pi_A^{\mathrm{lo}} \approx 0.396175$.

Other tie rules in the example. Take the prior $1/2$ and a slot with $\tau(r_1) < M(1/2) < \tau(r_2)$. The threshold pair has entry one after a purchase at r_A and is an equilibrium there under the default tie rules. When an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, the threshold pair is also an equilibrium on $(r_A, r_J]$, and its entry falls strictly from $\mathsf{E}_{\mathrm{R}}(r_A)$ at r_A to ρ at r_J, by Lemma A.10(b) and (c) and Proposition 3(c)(iv); at r_C, entry after a purchase in a full purchase can then take any value in $[\rho, 1]$. When an indifferent informed trader abstains, under either tie rule for the rival, the equilibrium set is an uninformative market on $[r_0, r_F]$; an uninformative market and a full purchase on $(r_F, r_N]$; empty on $(r_N, r_S]$, where the low class has no qualifying sale rate; and a full purchase on $(r_S, r_2]$.

At a general prior and any slot with $0 < \lambda < 1$ and $0 < \beta < 1/2$, when an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, the coexistence set is

$$\widetilde{C}(\pi) = C(\pi) \cup \{r_\pi\} \quad \text{for } \tau_A < \pi \le \pi_{\max}, \qquad \widetilde{C}(\pi) = C(\pi) \text{ otherwise.}$$

At $\pi = \pi_{\max}$, the point $r_\pi = r_* = 1.634218$ is the whole set. At the example's slot ($\lambda = 0.336122$, $\beta = 0.294655$), the top posterior is $M(\pi_{\max}) = 0.834698$, the posterior after the zero order is 0.454140, the informative market is a full purchase, which earns $\Delta_T(r_*)[1 - M(\pi_{\max})] - k = 0.000343$

net of the trading cost, and entry is 0.560586. At every prior and every slot, when an indifferent informed trader abstains and an indifferent high-cost rival prepares, coexistence holds on the strengths of $C(\pi)$ above $\mathfrak{r}\big(k/(1-M(\pi))\big)$; when an indifferent informed trader abstains and an indifferent high-cost rival may prepare with any probability, the point r_π joins them if $\mathfrak{r}\big(k/(1-M(\pi))\big) < r_\pi \leq r_N(\pi)$. Table A.3 records the dependence at the prior $1/2$.

Proof of the statements on other tie rules in the example. At the prior $1/2$, Steps 2 to 4 of the proof of Corollary 2 hold under each pair of tie rules. Under the default tie rules, Step 2 gives the threshold pair at r_A with $e^* = 1$, and Step 7 gives its entry $\mathsf{E}_\mathrm{R}(r_A)$. When an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, Step 2 gives the threshold pair on $[r_A, r_J]$, Step 4 gives $s_+ \geq s_-$, and Lemma A.9 completes the tuple. On the path the strengths of Proposition 3(c)(iv) are $[r_A, r_J]$: $\tau > 1/2$ everywhere, $\tau < M$ exactly below r_C, $r_J < r_C$, and $\rho J \leq k \leq J$ exactly on $[r_A, r_J]$. Lemma A.10(b) and (c) give the hypotheses of that part, and they give strict changes as well: $k - \rho J$ is positive on $[r_A, r_J)$, zero at r_J and strictly decreasing, and $\Delta_T(1-\tau)^2$ is positive and strictly increasing, so E_T falls strictly. At r_A, $e^* = 1$ gives $\mathsf{E}_\mathrm{T}(r_A) = \mathsf{E}_\mathrm{R}(r_A)$; at r_J, $\rho J = k$ gives $\mathsf{E}_\mathrm{T}(r_J) = \rho$. The full purchase at r_C is in Step 2. When an indifferent informed trader abstains, Step 2 removes the pairs that need a mixing informed trader and leaves the full purchase on $(r_F, r_2]$, and Step 3 leaves no low-class rate on $(r_N, r_S]$; the stated set follows.

At a general prior, the statements on other tie rules against a general distribution apply at each strength of the path. When an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, they add to $C(\pi)$ the strengths with $\tau(r) = \pi$ and $k/(1-\pi) < \Delta_T(r) \leq \Delta_T^\mathrm{N}$. Such a strength is r_π. There $k/(1-\pi) < \Delta_T(r_\pi)$ is $J(r_\pi) > k$, that is $r_\pi > r_A$, or $\pi > \tau_A$, by Lemma A.10(a) and (b); and for $\pi > \tau_A > 1/2$, $\Delta_T(r_\pi) \leq \Delta_T^\mathrm{N}$ is $\rho\Delta_T(r_\pi)\tau(r_\pi) \leq k$, that is $r_\pi \leq r_*$, or $\pi \leq \pi_\mathrm{max}$, by Lemma A.10(d). For $\tau_A < \pi \leq \pi_\mathrm{max}$ the root r_π exists, because $\tau(r_0) < \tau_A$ and $\tau(r_*) < \tau(r_2)$. At π_max, Corollary A.2(d) makes $C(\pi_\mathrm{max})$ empty, so the point r_* is the whole set. At the example's slot the four values follow from the construction at $\tau = \pi$, with the purchase rate, the posteriors and the entry formula of Proposition 3(a); the purchase is full because $\Delta_T(r_*)[1 - M(\pi_\mathrm{max})] > k$. When an indifferent informed trader abstains, the general statements give the strengths of $C(\pi)$ with $\Delta_T(r) > k/(1-M(\pi))$, that is $r > \mathfrak{r}\big(k/(1-M(\pi))\big)$, when the rival prepares; when it may prepare with any probability, r_π joins them if $k/(1-M(\pi)) < \Delta_T(r_\pi) \leq \Delta_T^\mathrm{N}$, that is $\mathfrak{r}\big(k/(1-M(\pi))\big) < r_\pi \leq r_N(\pi)$. □

When an indifferent informed trader abstains, a full purchase on $(r_N, r_S]$ is not an equilibrium, because the low class has no qualifying rate there (Lemma A.8). So the equilibrium set is nonempty along the whole path when an indifferent informed trader may mix, and not when it abstains; the strict rise of entry along a mixed purchase that recruits also uses the first rule. The tie rule for the

Table A.3. Which conclusions depend on the tie rules, at the prior $1/2$. The column heads give the choice of an indifferent informed trader, which may mix over its best orders or abstains, and of an indifferent high-cost rival, which prepares or may prepare with any probability.

	Informed trader may mix		Informed trader abstains	
Result	prepares	any probability	prepares	any probability
Proposition 2	holds	holds	holds	holds
Uninformative market **N**	$[r_0, r_N]$	$[r_0, r_N]$	$[r_0, r_N]$	$[r_0, r_N]$
Mixed purchase **R**	(r_A, r_F)	(r_A, r_F)	none	none
Threshold pair **T**	r_A only	$[r_A, r_J]$	none	none
Full purchase **F**	$[r_F, r_2]$	$[r_F, r_2]$	$(r_F, r_N] \cup (r_S, r_2]$	$(r_F, r_N] \cup (r_S, r_2]$
Set on (r_N, r_J)	**F**; **Q**	**F**; **Q**; **T**	empty	empty
Set at r_J	**F**	**F**; **T**	empty	empty
Set on $(r_J, r_S]$	**F**	**F**	empty	empty
Entry after a purchase at r_C	1	any in $[\rho, 1]$	1	any in $[\rho, 1]$
Coexistence	$[r_A, r_N]$	$[r_A, r_N]$	$(r_F, r_N]$	$(r_F, r_N]$

rival matters only through the threshold pair and the point r_C.

A.5 Research

Equilibrium with research. Market makers and the rival do not observe the research choice; they use a conjectured intensity $\hat{\lambda}$. An equilibrium with research is a tuple $(\hat{\lambda}, \sigma_H, \sigma_L, \mu, P, \bar{\mu}, \zeta)$, together with the bids. Conditions (E1)–(E6) of Definition A.1 hold at the intensity $\hat{\lambda}$. Condition (E5) holds also at $\hat{\lambda} = 0$: a deviation in research makes a success possible, and the order after it must be optimal. A seventh condition completes the definition.

(E7) *Research optimality.* Hold P and ζ at their values in the tuple. Then $\hat{\lambda}$ is the research choice of a best response among all joint deviations in research and orders.

A joint deviation changes the true order probabilities, but not the prices or the entry rule. With the research value Ψ below, the expected trading payoff after a success at the candidate's prices, Lemma A.12 shows that the best research choice is unique and equals $\min\{\bar{\lambda}, \Psi/a\}$. Condition (E7) therefore reads $\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\}$, and research never has a tie.

Setting. The classes satisfy the class ranking: $G_H \leq G_L$, both class values are at least the reserve $p > 0$, and both class means are finite. The incumbent distribution F is continuous, $\Delta_T = t_H - t_L > 0$, $\gamma = g_H - g_L > 0$, and $\tau = (c_H - g_L)/\gamma > 1/2$. The prior is $1/2$, the preparation cost is zero with probability $\rho \in (0, 1)$ and $c_H > 0$ otherwise, $k, a > 0$, $0 < \beta < 1/2$, and $0 < \bar{\lambda} < 1$. Research costs $(a/2)\lambda^2$, succeeds with probability λ, and is unobserved by market makers and the rival. At

a success the informed trader learns the class. The independence assumptions of the model hold. Write

$$b = (1-\lambda)\beta, \qquad b_0 = (1-\lambda)(1-2\beta).$$

$$m_\lambda = \frac{\beta(1-\lambda)}{2\beta + (1-2\beta)\lambda}, \quad M_\lambda = 1 - m_\lambda, \qquad m'_\lambda = -\frac{\beta}{[2\beta + (1-2\beta)\lambda]^2}.$$

For $1/2 < \tau < 1$, let

$$\lambda_E(\tau) = \frac{\beta(2\tau - 1)}{\beta + (1-2\beta)(1-\tau)}, \qquad J = \Delta_T(1-\tau).$$

For $\rho\Delta_T \geq 2k$, let $\lambda_Q(\Delta_T)$ be the unique root in $[0, 1)$ of

$$a\lambda_Q = \rho\Delta_T m_{\lambda_Q} - k. \tag{A.11}$$

It equals zero exactly when $\rho\Delta_T = 2k$. A path $(F_r)_{r\in I}$ holds every other primitive fixed and strengthens the incumbent in first-order stochastic dominance as r rises. The spread and, on $\tau > 1/2$, the preparation threshold are nondecreasing along such a path.

For a candidate with conjectured research $\hat{\lambda}$, write $\hat{b} = (1-\hat{\lambda})\beta$, $\hat{b}_0 = (1-\hat{\lambda})(1-2\beta)$, and $f_j(q) = (1-\hat{\lambda})\eta(q) + \hat{\lambda}\sigma_j(q)$. A candidate satisfies beliefs, competitive pricing, preparation and truthful bidding. Lemma A.3 gives its trading payoffs after a success:

$$U_H(q) = q\, e(q)\Delta_T[1-\mu(q)] - k|q|, \qquad U_L(q) = -q\, e(q)\Delta_T\mu(q) - k|q|. \tag{A.12}$$

The inverse posterior map satisfies $d\lambda_E/d\tau = \beta/[\beta + (1-2\beta)(1-\tau)]^2 > 0$.

Lemma A.12 (Joint deviations)**.** *For a candidate satisfying beliefs, competitive pricing, preparation and truthful bidding, let* $\Psi = \frac{1}{2}\max_q U_H(q) + \frac{1}{2}\max_q U_L(q)$ *be its best trading value after a research success. At its fixed prices and preparation rule, the best payoff of a joint deviation to any* $\tilde{\lambda} \in [0, \bar{\lambda}]$ *is* $\tilde{\lambda}\Psi - (a/2)\tilde{\lambda}^2$*. The unique best research choice is* $\min\{\bar{\lambda}, \Psi/a\}$*. Thus research optimality is exactly* $\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\}$*, under all four tie rules.*

Proof. A joint deviation changes the true order distribution to $(1-\tilde{\lambda})\eta + \tilde{\lambda}\sigma'_j$. It does not change the prices or the preparation rule, because market makers and the rival use the candidate's $\hat{\lambda}$ and σ_j. The informed trader trades only after a success, so its expected payoff is

$$\Pi(\tilde{\lambda}, \sigma') = \tilde{\lambda}\Big[\tfrac{1}{2}\sum_q \sigma'_H(q)U_H(q) + \tfrac{1}{2}\sum_q \sigma'_L(q)U_L(q)\Big] - \frac{a}{2}\tilde{\lambda}^2,$$

with U_j from (A.12) at the candidate's prices and entry. The bracket is at most Ψ, with equality exactly when each σ'_j puts weight only on maximizers of U_j. So $\sup_{\sigma'} \Pi(\tilde{\lambda}, \sigma') = \tilde{\lambda}\Psi - (a/2)\tilde{\lambda}^2$.

This function of $\tilde{\lambda}$ is strictly concave with derivative $\Psi - a\tilde{\lambda}$. If $\Psi \geq a\bar{\lambda}$, the derivative is positive on $[0, \bar{\lambda})$, and the maximizer is $\bar{\lambda}$. If $\Psi < a\bar{\lambda}$, the maximizer is $\Psi/a \in [0, \bar{\lambda})$. Strict concavity makes the maximizer unique, and it also excludes a nondegenerate mixture over research intensities. □

Lemma A.13 (Bounds under every profile)**.** *Let* $0 \leq \hat{\lambda} \leq \bar{\lambda} < 1$. *Under all four tie rules:*

(a) For every order profile, the nonzero-order posteriors lie in $[m_{\hat{\lambda}}, M_{\hat{\lambda}}]$, *with* $M_{\hat{\lambda}} \leq M_{\bar{\lambda}}$. *If also* $\beta \leq 1/3$, *the same bound holds for the zero-order posterior under every profile.*

(b) For every candidate satisfying beliefs, competitive pricing, preparation and truthful bidding, $P(q) = P(q')$ *implies* $\mu(q) = \mu(q')$, *and* $\bar{\mu}(P(q)) = \mu(q)$.

(c) Wrong-signed orders satisfy $U_H(-1) \leq -k$ *and* $U_L(1) \leq -k$; *every equilibrium has* $\sigma_H(-1) = \sigma_L(1) = 0$.

(d) With no wrong-signed orders, $\mu(-1) \leq 1/2 \leq \mu(1)$. *If also* $\sigma_H(1) = 1$, *then* $\mu(1) = M_{\hat{\lambda}}$ *and* $\mu(0) \leq 1/2$.

Proof. (a) At each nonzero order, its class likelihood lies between $\hat{b}$ and $\hat{b} + \hat{\lambda}$. Bayes' rule at the prior $1/2$ therefore puts its posterior between $\hat{b}/(2\hat{b} + \hat{\lambda}) = m_{\hat{\lambda}}$ and $M_{\hat{\lambda}}$. At zero the same argument uses $\hat{b}_0$; its bounds lie inside this interval when $\hat{b}_0 \geq \hat{b}$, or $\beta \leq 1/3$. The top bound rises with research, so $M_{\hat{\lambda}} \leq M_{\bar{\lambda}}$.

(b) Orders with the same price have the same preparation probability and hence the same entry. Competitive pricing gives $P(q) = t_0 + e(q)[t_L - t_0 + \Delta_T \mu(q)]$. Since $e(q) \geq \rho > 0$ and $\Delta_T > 0$, equal prices then imply equal posteriors. Averaging the posteriors at a common price gives that same posterior as the rival's belief.

(c) The two wrong-signed payoffs in (A.12) are at most $-k$, while abstention earns zero. They therefore receive zero probability under either tie rule for the informed trader.

(d) With no wrong-signed orders, $f_H(1) = \hat{b} + \hat{\lambda}\sigma_H(1) \geq \hat{b} = f_L(1)$ and $f_H(-1) = \hat{b} \leq \hat{b} + \hat{\lambda}\sigma_L(-1) = f_L(-1)$. With a full high-class purchase, $f_H(0) = \hat{b}_0 \leq \hat{b}_0 + \hat{\lambda}\sigma_L(0) = f_L(0)$ and $\mu(1) = (\hat{b} + \hat{\lambda})/(2\hat{b} + \hat{\lambda}) = M_{\hat{\lambda}}$. Bayes' rule gives the stated comparisons with the prior. □

Lemma A.14 (Positive research needs a strict full purchase)**.** *Under either tie rule for the informed trader and either tie rule for the rival, every equilibrium with* $\hat{\lambda} > 0$ *has* $U_H(1) > 0$ *and* $\sigma_H(1) = 1$. *Its posteriors satisfy* $\mu(1) = M_{\hat{\lambda}}$, $\mu(0) \leq 1/2$ *and* $\mu(-1) \leq 1/2$; *entry after a sale or no order is* ρ. *This holds for every* $0 < \beta < 1/2$.

Proof. By Lemma A.12, $\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\} > 0$, so $\Psi > 0$. By Lemma A.13(c) and $U_j(0) = 0$, either $U_H(1) > 0$ or $U_L(-1) > 0$. Suppose $U_L(-1) > 0$. Then the sale is the unique maximizer of class L, and (E5) gives $\sigma_L(-1) = 1$. With $\sigma_H(-1) = 0$, $f_H(-1) = \hat{b}$ and $f_L(-1) = \hat{b} + \hat{\lambda}$, so

$\mu(-1) = m_{\hat{\lambda}} \le 1/2 < \tau$. Hence $e(-1) = \rho$ and $U_L(-1) = \rho\Delta_T m_{\hat{\lambda}} - k > 0$. The purchase then earns $U_H(1) = e(1)\Delta_T[1 - \mu(1)] - k \ge \rho\Delta_T(1 - M_{\hat{\lambda}}) - k = \rho\Delta_T m_{\hat{\lambda}} - k > 0$, by Lemma A.13(a). So $U_H(1) > 0$ in both cases. The purchase is the unique maximizer of class H, and $\sigma_H(1) = 1$ under both tie rules for the informed trader. Lemma A.13(d) gives the three posteriors. Since $\tau > 1/2$, Lemma A.2(d) gives $\zeta(P(0)) = \zeta(P(-1)) = 0$ under both tie rules for the rival. □

Lemma A.15 (The low class's sale)**.** *Let $0 < \lambda \le \bar{\lambda}$, let class H buy with probability one after a success, and suppose an indifferent informed trader may mix over its best orders. By Lemma A.8 at presence λ and prior $1/2$, the unique low-class sale rate consistent with its best orders is*

$$s_- = \lambda\sigma_L(-1) = \min\{\lambda,\ b(\rho\Delta_T/k - 2)^+\}. \tag{A.13}$$

The remaining low-class probability is on abstention, and

$$\mu(-1) = \frac{b}{2b + s_-}, \qquad \mu(0) = \frac{b_0}{2b_0 + \lambda - s_-}, \qquad \mu(1) = M_\lambda.$$

Both nonpurchase orders have entry ρ, under either tie rule for the rival. Class L abstains when $\rho\Delta_T \le 2k$; it mixes when $\rho\Delta_T m_\lambda < k < \rho\Delta_T/2$, with $\mu(-1) = k/(\rho\Delta_T)$; and it sells with probability one when $\rho\Delta_T m_\lambda \ge k$. At $\rho\Delta_T = 2k$ its sale rate is zero; at $\rho\Delta_T m_\lambda = k$ its sale probability is one. Its best net payoff is

$$\max_q U_L(q) = (\rho\Delta_T m_\lambda - k)^+.$$

Proof. Lemma A.8, at presence λ and prior $1/2$, gives (A.13). In order $(-1, 0, 1)$, the class likelihoods are

$$f_H = (b, b_0, b + \lambda), \qquad f_L = (b + s_-, b_0 + \lambda - s_-, b).$$

Bayes' rule gives the displayed posteriors. Both nonpurchase posteriors are at most $1/2 < \tau$, so entry there is ρ. The sale payoff

$$U_L(-1) = \frac{\rho\Delta_T b}{2b + s_-} - k$$

falls strictly in s_-. At zero sale rate it is nonpositive exactly when $\rho\Delta_T \le 2k$; at full selling it is nonnegative exactly when $\rho\Delta_T m_\lambda \ge k$. Between these cases its unique zero gives the mixing rate and the posterior $k/(\rho\Delta_T)$. At the first equality any positive rate makes selling lose; at the second equality any smaller rate makes selling strictly profitable. This proves both boundary choices. During abstention or mixing the best low-class payoff is zero; at full selling it is $\rho\Delta_T m_\lambda - k$, which gives its positive-part expression. □

Proposition 4 (Competition creates research, full statement). *Maintain the setting above. An indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares.*

When an indifferent high-cost rival may prepare with any probability, select full preparation after a purchase that reaches the threshold.

(a) Recruiting research, an equilibrium with positive research and full preparation after a purchase, exists exactly when

$$\Delta_T > 2k, \qquad \frac{1}{2} < \tau \le M_{\min\{\lambda_R(\Delta_T),\bar{\lambda}\}}, \tag{A.14}$$

where $\lambda_R(\Delta_T)$ is the unique root in $(0,1)$ of

$$2a\lambda = \Delta_T m_\lambda - k + (\rho\Delta_T m_\lambda - k)^+. \tag{A.15}$$

Its research is $\hat{\lambda} = \min\{\lambda_R(\Delta_T), \bar{\lambda}\}$. Class H buys strictly and fully, class L uses Lemma A.15, and entry is $e(1) = 1$, $e(0) = e(-1) = \rho$. The posteriors are those of that lemma, prices are $P(x) = t_0 + e(x)[t_L - t_0 + \Delta_T\mu(x)]$, price beliefs are $\bar{\mu}(P(x)) = \mu(x)$, and bids are truthful. Equal nonpurchase posteriors have equal entry and equal prices.

(b) Research is continuous and strictly increasing in Δ_T below its cap, and constant at the cap. Total entry and class entry are

$$\begin{aligned} \mathsf{E} &= \rho + (1-\rho)[\beta + (1/2 - \beta)\hat{\lambda}], \\ e_H &= \rho + (1-\rho)[\beta + (1-\beta)\hat{\lambda}], \qquad e_L = \rho + (1-\rho)\beta(1-\hat{\lambda}). \end{aligned}$$

Thus total entry and class-H entry rise, class-L entry falls, and the class-H share of prepared rivals rises with the spread, strictly below the cap and weakly throughout the existence set. The prepared-value distribution $(e_H G_H + e_L G_L)/(e_H + e_L)$ improves in first-order stochastic dominance; its mean rises strictly when research rises strictly. For almost every fixed incumbent value x, the high-class acquisition probability is

$$\Pr(H \text{ acquires} \mid R = x) = \begin{cases} e_H/2, & x < p, \\ e_H[1 - G_H(x)]/2, & x \ge p. \end{cases}$$

It rises strictly with the spread below the cap wherever class H can win. At a class-value atom the auction's tie allocation gives the probability of winning; the same comparison holds using that fixed allocation. At a value outside the incumbent's support, this expression gives the counterfactual allocation at that value.

While the low class abstains or mixes, and after a sure sale becomes strictly profitable, respectively,

$$\frac{d\lambda_R}{d\Delta_T} = \frac{m_{\lambda_R}}{2a - \Delta_T m'_{\lambda_R}}, \qquad \frac{d\lambda_R}{d\Delta_T} = \frac{(1+\rho)m_{\lambda_R}}{2a - (1+\rho)\Delta_T m'_{\lambda_R}}.$$

If $(1-\rho)k/(2a\rho) < \bar{\lambda}$*, these slopes meet with a strict upward jump at* $\lambda_R = (1-\rho)k/(2a\rho)$*. On an equilibrium path that passes through this intensity, a sure sale's profit adds a second return to research. Research and entry are continuous through the first sale, the onset of strictly profitable full selling, and the cap. The cap spread, at which research reaches its cap, is*

$$\begin{cases} (2a\bar{\lambda}+k)/m_{\bar{\lambda}}, & \bar{\lambda} \le (1-\rho)k/(2a\rho), \\ (2a\bar{\lambda}+2k)/[(1+\rho)m_{\bar{\lambda}}], & \bar{\lambda} > (1-\rho)k/(2a\rho). \end{cases}$$

(c) *High-class ownership is* $\mathsf{O}_H(F) = e_H(F)W_H(F)/2$*. For two recruiting economies on a path,*

$$\mathsf{O}_H(F_{r'}) \ge \mathsf{O}_H(F_r) \quad \Longleftrightarrow \quad \frac{W_H(F_{r'})}{W_H(F_r)} \ge \frac{e_H(F_r)}{e_H(F_{r'})},$$

with the corresponding strict equivalence. These win probabilities are positive. If class H outbids every incumbent along the recruiting part of the path, ownership rises weakly with strength, and strictly for a strict increase in the spread from a research intensity below the cap. Decisive advantage (Lemma A.6), $g_H(F) > c_H \Longrightarrow W_H(F) = 1$ *throughout the family, supplies this outbidding condition at every recruiting economy.*

(d) *Let* $I = [r_-, r_+]$ *be compact and* Δ_T, τ *continuous along the path. The capped recruiting set is*

$$\{r \in I : \Delta_T(r)m_{\bar{\lambda}} - k + (\rho\Delta_T(r)m_{\bar{\lambda}} - k)^+ \ge 2a\bar{\lambda},\ \tau(r) \le M_{\bar{\lambda}}\}.$$

It is empty, a singleton, or a nondegenerate closed interval. If nonempty, it ends at $\max\{r \in I : \tau(r) \le M_{\bar{\lambda}}\}$.

Suppose the path first reaches the cap spread at $r_{\bar{\lambda}} > r_-$*, with a strictly smaller spread at* r_-*. On* $[r_-, r_{\bar{\lambda}}]$*, suppose* $\tau < 1$*, and define*

$$\varphi_a^* = J - k + (\rho J - k)^+ - 2a\lambda_E(\tau).$$

If φ_a^* *is strictly increasing on* $[r_-, r_{\bar{\lambda}}]$ *and* $\varphi_a^*(r_-) < 0 < \varphi_a^*(r_{\bar{\lambda}})$*, the full recruiting set is*

$$[r_R,\ \max\{r \in I : \tau(r) \le M_{\bar{\lambda}}\}],$$

where r_R *is the unique zero of* φ_a^* *before* $r_{\bar{\lambda}}$*. Research is below its cap up to* $r_{\bar{\lambda}}$ *and at the cap thereafter. A threshold plateau at* $M_{\bar{\lambda}}$ *belongs to the recruiting set through its last point. On a*

path with continuously differentiable Δ_T, τ on $[r_-, r_{\bar\lambda}]$, the cheap-research condition

$$[1 + \rho \mathbf{1}\{\rho J > k\}]J' - \frac{2a\beta\tau'}{[\beta + (1-2\beta)(1-\tau)]^2} > 0 \quad \textit{almost everywhere on } [r_-, r_{\bar\lambda}] \tag{CR}$$

makes φ_a^* strictly increasing there.

Proof. Part (a). Positive research forces the strict full purchase in Lemma A.14. With full preparation after that purchase, its net payoff is $\Delta_T m_\lambda - k$, and Lemma A.15 gives the other class's best payoff. Thus

$$\Psi = \tfrac{1}{2}\left[\Delta_T m_\lambda - k + (\rho \Delta_T m_\lambda - k)^+\right].$$

Lemma A.12 gives (A.15) below the cap and its weak inequality at the cap. The residual

$$\Delta_T m_\lambda - k + (\rho \Delta_T m_\lambda - k)^+ - 2a\lambda$$

is continuous and strictly decreasing in λ. If $\Delta_T > 2k$, it is positive at zero and negative at one, so it has exactly one root in $(0, 1)$. If $\Delta_T \le 2k$, every positive research intensity makes $\Delta_T m_\lambda - k < 0$, contrary to the strict purchase. This proves the root condition and the capped research choice. At the capped choice the residual is nonnegative. A nonpositive purchase payoff would make the sale's positive part zero, contradicting $2a\hat\lambda > 0$; the purchase is therefore strict.

A purchase recruits exactly when $M_{\hat\lambda} \ge \tau$, including equality under the stated preparation choice. This gives (A.14). Conversely, under that condition use the sale lemma's likelihoods and posteriors, the stated entry, and the competitive prices. For distinct posteriors $\mu(x) > \mu(y)$, entry is nondecreasing in the posterior and

$$P(x) - P(y) = e(x)\Delta_T[\mu(x) - \mu(y)] + [e(x) - e(y)][t_L - t_0 + \Delta_T \mu(y)] > 0,$$

since $t_L \ge t_0$ and $e(x) \ge \rho > 0$. Equal nonpurchase posteriors have equal entry ρ and equal prices, so each price belief equals the posterior of its orders. The threshold rule gives preparation, the strict purchase and the sale lemma give optimal orders, and truthful bids give the auction allocation. The research equality or cap inequality gives optimality of every joint deviation by Lemma A.12. These conditions also determine the positive-research tuple uniquely.

Part (b). For a fixed positive intensity, the residual rises strictly with Δ_T. Its strict decrease in intensity makes its root continuous and strictly increasing in the spread. Clipping the root at $\bar\lambda$ preserves continuity and gives constancy at the cap. A purchase has probability $\frac{1}{2}(b + \hat\lambda) + \frac{1}{2}b = \beta + (1/2 - \beta)\hat\lambda$. Only a purchase raises entry above ρ, which gives E. Conditional purchase probabilities $b + \hat\lambda$ and b give e_H and e_L. Their derivatives in research have signs positive and

negative, respectively, and

$$\frac{d}{d\hat{\lambda}}\frac{e_H}{e_H+e_L}=\frac{(1-\rho)[\rho+(1-\rho)\beta]}{(e_H+e_L)^2}>0.$$

Preparation is independent of values conditional on class. An increase in this share therefore shifts the prepared-value distribution toward G_H, which dominates G_L. Its mean rises strictly because $\mathbb{E}_H\theta-\mathbb{E}_L\theta=\Delta_T+\gamma>0$. The same independence gives the displayed acquisition probability at a fixed incumbent value; its win factor is fixed by the class distribution and the auction's allocation at a tie, so its change has the sign of the change in e_H wherever that factor is positive.

Implicit differentiation of the two smooth pieces of (A.15) gives the two slopes in the statement. At the sale-profit boundary, $\rho\Delta_T m_{\lambda_R}=k$; combining this with $2a\lambda_R=\Delta_T m_{\lambda_R}-k$ gives $\lambda_R=(1-\rho)k/(2a\rho)$. At the common point, subtracting the first slope from the second gives

$$\frac{2a\rho m_{\lambda_R}}{(2a-\Delta_T m'_{\lambda_R})[2a-(1+\rho)\Delta_T m'_{\lambda_R}]}>0.$$

The sale rate and its positive-part payoff are continuous at both sale boundaries. The research root and entry formulas are continuous there and at the cap. Substituting $\lambda=\bar{\lambda}$ in the research equation on each side of the sale-profit boundary gives the stated cap spread.

Part (c). Conditional independence gives $\mathsf{O}_H(F)=\Pr(H)\Pr(\text{preparation}\mid H)W_H(F)=e_H(F)W_H(F)/2$. At a recruiting economy, $c_H\le g_L+M_{\hat{\lambda}}\gamma<g_H$; hence $g_H>0$ and $W_H(F)>0$. Dividing the two positive ownership expressions proves the weak and strict equivalences. If class H outbids every incumbent, $W_H=1$, and part (b) gives the stated monotonicity. The inequality $g_H>c_H$ also makes the implication in decisive advantage apply.

Part (d), research at the cap. At a positive research intensity, class H buys strictly and fully. A sale and no order have posteriors at most $1/2$, and hence entry ρ. With full preparation after a purchase, the best trading payoffs are $\Delta_T m_\lambda-k$ in class H and $(\rho\Delta_T m_\lambda-k)^+$ in class L. Holding prices and the preparation rule fixed, a joint deviation to $\tilde{\lambda}$ yields

$$\frac{\tilde{\lambda}}{2}\left[\Delta_T m_\lambda-k+(\rho\Delta_T m_\lambda-k)^+\right]-\frac{a}{2}\tilde{\lambda}^2.$$

Its maximum over $[0,\bar{\lambda}]$ is at the cap exactly when

$$\Delta_T m_{\bar{\lambda}}-k+(\rho\Delta_T m_{\bar{\lambda}}-k)^+\ge 2a\bar{\lambda}.$$

The left side is continuous and strictly increasing in Δ_T. Equality has exactly one solution. If its sale term is zero, that solution is $(2a\bar{\lambda}+k)/m_{\bar{\lambda}}$, and the sale term is zero exactly when $\rho(2a\bar{\lambda}+k)\le k$,

or $\bar{\lambda} \le (1-\rho)k/(2a\rho)$. Otherwise equality gives the second displayed value of the cap spread. Both pieces agree at their common boundary. Since $m_{\bar{\lambda}} < 1/2$, the residual at $\Delta_T = 2k$ is strictly negative, and the cap spread exceeds $2k$.

At or above the cap spread, the high-class purchase earns strictly more than zero: a nonpositive purchase payoff would make the sale's positive part zero and contradict the positive right side $2a\bar{\lambda}$. The low class uses the unique sale rate

$$s_- = \min\{\bar{\lambda},\ \bar{b}(\rho\Delta_T/k - 2)^+\}, \qquad \bar{b} = (1-\bar{\lambda})\beta.$$

The purchase posterior is $M_{\bar{\lambda}}$. It recruits exactly when $\tau \le M_{\bar{\lambda}}$, with full preparation at equality under the stated rule. Bayes' rule, competitive prices and truthful bids give the remaining equilibrium conditions. In particular, if a sale and no order have the same posterior, they both have entry ρ and the same price, so the belief at that price is their common posterior. This proves the pointwise description.

Along the path, Δ_T is nondecreasing by Lemma A.5. Each class's gross profit and their positive gap are nonincreasing; since $c_H - g_L > 0$, $\tau = (c_H - g_L)/(g_H - g_L)$ is nondecreasing too. Consequently the set of strengths at or above the cap spread contains every strength above any of its members, and $\{\tau \le M_{\bar{\lambda}}\}$ contains every strength below any of its members. Continuity and compactness make both sets closed. Their intersection is empty, a singleton, or the interval with the endpoints in the statement. A stretch on which $\tau = M_{\bar{\lambda}}$ belongs to this interval whenever research has reached the cap. If the right endpoint lies in the interior of I, continuity gives $\tau = M_{\bar{\lambda}}$ there; if the path ends first, its last strength is the endpoint.

Part (d), the recruiting interval. For a fixed $\Delta_T > 2k$, the function

$$\Delta_T m_\lambda - k + (\rho\Delta_T m_\lambda - k)^+ - 2a\lambda$$

is continuous and strictly decreasing in λ, positive at zero and negative at one. Its unique zero is $\lambda_R(\Delta_T)$. At $\lambda = \lambda_E(\tau)$, the function equals φ_a^*, because $m_{\lambda_E} = 1 - \tau$. Therefore, when $1/2 < \tau < 1$, the purchase posterior reaches the threshold at the capped root exactly when

$$\varphi_a^* \ge 0 \quad \text{and} \quad \lambda_E(\tau) \le \bar{\lambda}.$$

A nonnegative margin implies $J - k + (\rho J - k)^+ \ge 2a\lambda_E > 0$. If $J \le k$, its left side is nonpositive, so $J > k$, which gives $\Delta_T > 2k$ since $1 - \tau < 1/2$. Thus these two inequalities also imply that the root exists. Conversely, the root comparison proves their necessity.

Continuity, strict increase and the endpoint signs of φ_a^* yield exactly one zero r_R, with a negative margin below it and a positive margin above it.

At $r_{\bar\lambda}$, continuity makes the spread equal to the cap spread, so $\lambda_R = \bar\lambda$. The positive margin there gives $\lambda_E(\tau(r_{\bar\lambda})) < \bar\lambda$, or $\tau(r_{\bar\lambda}) < M_{\bar\lambda}$. Since τ is nondecreasing, the cap posterior reaches the threshold throughout $[r_-, r_{\bar\lambda}]$. On that interval, the exact root comparison therefore makes recruitment equivalent to $\varphi_a^* \geq 0$, or $r \geq r_R$. Below $r_{\bar\lambda}$, the spread is strictly below the cap spread, so research is below its cap. From $r_{\bar\lambda}$ on, research optimality at the cap persists because Δ_T is nondecreasing. The cap result gives exactly the upper interval in the statement, including its last point and every threshold plateau within it.

When Δ_T, τ are continuously differentiable, φ_a^* is absolutely continuous on $[r_-, r_{\bar\lambda}]$. Away from $\rho J = k$ its derivative is the expression in (CR). On the level set $\rho J = k$, $J' = 0$ almost everywhere, so that expression is its derivative almost everywhere there as well. Integrating the strictly positive derivative over any nondegenerate subinterval proves that φ_a^* rises strictly. □

The research margin. Where $\tau' > 0$, condition (CR) says that

$$a < \frac{[1 + \rho \mathbf{1}\{\rho J > k\}]J'}{2\lambda_E'(\tau)\tau'}.$$

Where $\tau' = 0$, it says $J' > 0$. Before a sure sale pays, the derivative of the margin is $J' - 2a\lambda_E'(\tau)\tau'$; once it pays, the derivative is $(1+\rho)J' - 2a\lambda_E'(\tau)\tau'$. The margin is continuous at $\rho J = k$; at a crossing into $\rho J > k$, its derivative jumps up by $\rho J'$. The condition applies where the recruiting interval starts with a strictly profitable sure sale.

Ownership at the same incumbent. Where recruiting research coexists with zero research, $\rho\Delta_T \leq 2k$, their same-incumbent ownership ratio is $\mathsf{O}_H^{\mathrm{R}}/\mathsf{O}_H^{\mathrm{Z}} = e_H/\rho > 1$, which rises strictly with the spread below the research cap and stays constant at the cap.

Entry ρ after every order in Lemma A.16(a) gives $\mathsf{O}_H^{\mathrm{Z}} = \rho W_H(F)/2$. Dividing the recruiting ownership formula by it cancels the same positive win probability; the formula for e_H in Proposition 4(b) gives the strict increase below the cap and constancy at the cap.

Lemma A.16 (The three research outcomes)**.** *Maintain the setting above. An indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares. The following outcomes exhaust the equilibria.*

(a) Zero research exists exactly when $\rho\Delta_T \leq 2k$. All posteriors are $1/2$, entry is ρ, and every order has price $t_0 + \rho[t_L - t_0 + \Delta_T/2]$. If the inequality is strict, both classes abstain after a hypothetical success. At equality, each class uses any mixture of its correct order and abstention; all these strategies give the same zero-research outcome.

(b) Recruiting research is exactly the outcome of Proposition 4(a).

(c) Floor-entry research exists exactly when

$$\rho\Delta_T > 2k, \qquad \tau > M_{\min\{\lambda_Q(\Delta_T),\bar{\lambda}\}},$$

where λ_Q is the root of (A.11). Research is $\hat{\lambda} = \min\{\lambda_Q, \bar{\lambda}\}$, both classes trade strictly and fully after a success, the posteriors are $(m_{\hat{\lambda}}, 1/2, M_{\hat{\lambda}})$, and entry is ρ after every order. Prices are $t_0 + \rho[t_L - t_0 + \Delta_T\mu(x)]$, price beliefs are the corresponding posteriors, and bids are truthful.

Each positive-research outcome has a unique tuple. The two positive-research outcomes coexist exactly when

$$\rho\Delta_T > 2k, \qquad M_{\min\{\lambda_Q,\bar{\lambda}\}} < \tau \leq M_{\min\{\lambda_R,\bar{\lambda}\}}.$$

The uncapped roots satisfy $\lambda_Q < \lambda_R$; their capped values are strictly ordered when $\lambda_Q < \bar{\lambda}$ and coincide when both roots reach the cap. Thus the coexistence band has positive width in τ exactly when $\lambda_Q < \bar{\lambda}$. At least one of the three outcomes exists at every auction in the stated domain, including $\tau \geq 1$.

Proof. At conjectured research zero, $f_j(x) = \eta(x) > 0$ for both classes, regardless of their orders after a hypothetical success. Bayes' rule gives the posterior $1/2$ at every order. A belief at a price averages these identical posteriors, so it too is $1/2$, even if several orders share the price. Since $\tau > 1/2$, the high-cost rival strictly prefers abstention after each price, under either tie rule for the rival. Entry is therefore ρ. Competitive pricing then gives the common price in the statement. This establishes beliefs, preparation and prices without any restriction on the hypothetical-success orders.

At these fixed prices and entry, the correct-order payoffs are

$$U_H(1) = U_L(-1) = \rho\Delta_T/2 - k,$$

while $U_H(0) = U_L(0) = 0$, and the wrong-signed payoffs are $U_H(-1) = U_L(1) = -\rho\Delta_T/2 - k < 0$. The best trading value after a success is therefore $\Psi = (\rho\Delta_T/2 - k)^+$. A joint deviation to $\tilde{\lambda} \in [0, \bar{\lambda}]$, with optimal orders at those prices and entry, gives

$$\tilde{\lambda}\Psi - \frac{a}{2}\tilde{\lambda}^2.$$

If $\rho\Delta_T \leq 2k$, then $\Psi = 0$, and the unique optimal research choice is zero. If $\rho\Delta_T > 2k$, then $\Psi > 0$, and the unique optimum $\min\{\bar{\lambda}, \Psi/a\}$ is positive, so zero research fails research optimality. This proves necessity and sufficiency for research, including all joint order deviations.

When $\rho\Delta_T < 2k$, both correct orders are strictly worse than abstention, which gives the unique pair of hypothetical-success distributions. At equality, the high class is indifferent exactly between

buying and abstaining, and the low class exactly between selling and abstaining. Any mixtures over these best orders obey order optimality when an indifferent informed trader may mix. They leave $f_j = \eta$, beliefs, prices and entry unchanged because their occurrence probability is zero. Truthful bids complete each equilibrium. Preparation is independent of values conditional on class, so high-class ownership is $\rho W_H(F)/2$. The abstention rule selects the point masses on zero from the same best-order sets.

When class H buys and class L sells after a success and no order recruits, entry is ρ after every order, and the purchase and the sale both earn $\rho\Delta_T m_\lambda - k$ at the conjectured research λ. For $\rho\Delta_T \geq 2k$, the root $\lambda_Q(\Delta_T)$ of (A.11) exists and is unique, because the gap $\rho\Delta_T m_\lambda - k - a\lambda$ falls strictly in λ, from $\rho\Delta_T/2 - k \geq 0$ at $\lambda = 0$ to $-k - a$ at $\lambda = 1$. The root is zero at $\rho\Delta_T = 2k$ and positive when $\rho\Delta_T > 2k$. The gap rises strictly in Δ_T, and its derivative in λ is negative, so λ_Q rises strictly and continuously in Δ_T.

Under an indifferent informed trader who may mix and an indifferent high-cost rival who prepares, take an equilibrium with positive research. Class H buys strictly and fully, so $\mu(1) = M_\lambda$ and $e(0) = e(-1) = \rho$. If $M_\lambda \geq \tau$, the purchase recruits fully and the sale rule and research condition give recruiting research, with $\lambda = \min\{\lambda_R(\Delta_T), \bar{\lambda}\}$. If $M_\lambda < \tau$, the strict purchase payoff is $\rho\Delta_T m_\lambda - k > 0$. Any low-class sale payoff is at least this, because $\mu(-1) \geq m_\lambda$. Thus class L sells strictly and fully, both best trading payoffs equal $\rho\Delta_T m_\lambda - k$, and research is $\min\{\lambda_Q(\Delta_T), \bar{\lambda}\}$, with λ_Q from (A.11). This root is positive exactly when $\rho\Delta_T > 2k$. At equality, every positive λ has $\rho\Delta_T m_\lambda - k < 0$, so floor-entry research has no positive-intensity endpoint there.

For $\rho\Delta_T > 2k$, the recruiting residual at λ_Q equals

$$\Delta_T m_{\lambda_Q} - k + (\rho\Delta_T m_{\lambda_Q} - k)^+ - 2a\lambda_Q = (1-\rho)\Delta_T m_{\lambda_Q} > 0.$$

Its strict decrease yields $\lambda_Q < \lambda_R$, hence $\min\{\lambda_Q, \bar{\lambda}\} \leq \min\{\lambda_R, \bar{\lambda}\}$, with strict inequality whenever $\lambda_Q < \bar{\lambda}$. The two positive-research outcomes coexist exactly when

$$\rho\Delta_T > 2k, \qquad M_{\min\{\lambda_Q,\bar{\lambda}\}} < \tau \leq M_{\min\{\lambda_R,\bar{\lambda}\}}.$$

If both roots reach the cap, this interval is empty. Otherwise it has positive width in the threshold coordinate. Together with the zero-research case, this exhausts the outcomes. Each positive-research outcome fixes its research, orders, posteriors, competitive prices, preparation probabilities and truthful bids; at the zero-research boundary, the best-order mixtures above give all the hypothetical-success strategies.

For the floor-entry construction, its research choice makes $\rho\Delta_T m_{\hat{\lambda}} - k \geq a\hat{\lambda} > 0$, with equality below the cap. Both correct orders are therefore strictly best. Full orders give the three stated posteriors, and constant entry ρ makes the three prices distinct. The strict threshold inequality

gives preparation; competitive prices, their posterior beliefs and truthful bids give the remaining conditions. The research choice follows from Lemma A.12. If $\rho\Delta_T \le 2k$, zero research exists. Otherwise the ordering of the capped roots implies that either $\tau \le M_{\min\{\lambda_R,\bar{\lambda}\}}$ and recruitment exists, or $\tau > M_{\min\{\lambda_R,\bar{\lambda}\}} \ge M_{\min\{\lambda_Q,\bar{\lambda}\}}$ and floor-entry research exists. This also covers $\tau \ge 1$. □

Lemma A.17 (Strong incumbents)**.** *Maintain the setting above. If $M_{\bar{\lambda}} \neq \tau$ and*

$$\rho\Delta_T m_{\bar{\lambda}} - k > a\bar{\lambda}, \tag{U}$$

then under all four tie rules the equilibrium is unique. Research is $\bar{\lambda}$, class H buys and class L sells strictly after a success, and the posteriors are $(m_{\bar{\lambda}}, 1/2, M_{\bar{\lambda}})$. Entry after a sale and no order is ρ, while $\zeta(P(1)) = \mathbf{1}\{M_{\bar{\lambda}} > \tau\}$. Competitive prices, their posterior beliefs and truthful bids complete the tuple. With $\bar{b} = (1-\bar{\lambda})\beta$, total entry is $\rho + (1-\rho)(\bar{b} + \bar{\lambda}/2)$ if $M_{\bar{\lambda}} > \tau$ and ρ otherwise.

Proof. Uniqueness. Take any candidate. By Lemma A.13(a), every nonzero-order posterior lies in $[m_{\bar{\lambda}}, M_{\bar{\lambda}}]$ whatever the orders, and $e \ge \rho$ after every price. Hence $U_H(1) \ge \rho\Delta_T(1 - M_{\bar{\lambda}}) - k$ and $U_L(-1) \ge \rho\Delta_T m_{\bar{\lambda}} - k$. Both lower bounds equal $\rho\Delta_T m_{\bar{\lambda}} - k > a\bar{\lambda} \ge 0$ by (U). So the purchase and the sale are the strict best orders after a success, under both tie rules for the informed trader. Then $\Psi = \frac{1}{2}[U_H(1) + U_L(-1)] > a\bar{\lambda}$, and Lemma A.12 gives $\hat{\lambda} = \bar{\lambda}$. Write $\bar{b}_0 = (1-\bar{\lambda})(1-2\beta)$. Full orders give $f_H = (\bar{b}, \bar{b}_0, \bar{b} + \bar{\lambda})$ and $f_L = (\bar{b} + \bar{\lambda}, \bar{b}_0, \bar{b})$, so the posteriors are $(m_{\bar{\lambda}}, 1/2, M_{\bar{\lambda}})$. Since $\tau > 1/2$, $\zeta(P(0)) = \zeta(P(-1)) = 0$. Since $M_{\bar{\lambda}} \neq \tau$, $\zeta(P(1))$ is 1 or 0 with no tie, under both tie rules for the rival. Every component of the tuple is fixed.

Existence. (E1): the likelihoods above. (E2): the stated entry is nondecreasing in the posterior, so the three prices differ, and each price reveals its order. (E3): $B(M_{\bar{\lambda}})$ lies strictly on one side of c_H, and $B(1/2), B(m_{\bar{\lambda}}) < c_H$. (E4): (A.4). (E5): the strict bounds above, read at the tuple. (E6): truthful bids. (E7): $\Psi > a\bar{\lambda}$. The purchase has probability $\frac{1}{2}(\bar{b} + \bar{\lambda}) + \frac{1}{2}\bar{b} = \bar{b} + \bar{\lambda}/2$, and it recruits exactly when $M_{\bar{\lambda}} > \tau$. □

Access to prices with research. Maintain the general setting, condition (U), and $\tau < M_{\bar{\lambda}}$. Under all four tie rules, both the economy with price access and the economy in which the rival does not see the price have full orders and research at the cap, uniquely. With access only a purchase recruits; with the price hidden entry is ρ after every order. Hence access raises expected proceeds by

$$(1-\rho)(\bar{b} + \bar{\lambda}/2)[t_L - t_0 + \Delta_T M_{\bar{\lambda}}] > 0$$

and surplus, net of research, trading and preparation costs, by

$$(1-\rho)(\bar{b} + \bar{\lambda}/2)[\gamma(M_{\bar{\lambda}} - \tau) + pF(p)] > 0.$$

Proof. With the price hidden, the rival retains its prior $1/2 < \tau$, so entry is ρ after every order. The nonzero-order bounds in Lemma A.17 apply with this constant entry, and condition (U) forces full orders and research at the cap, uniquely. With access the same lemma gives full orders, cap research and recruitment only after a purchase. Thus the order distribution, research cost $(a/2)\bar{\lambda}^2$ and trading cost $k\bar{\lambda}$ coincide. A purchase has probability $\bar{b} + \bar{\lambda}/2$, with conditional high-class probability $M_{\bar{\lambda}}$. Multiplying its additional preparation probability by its conditional proceeds gain gives the first display. For any value $\theta \geq p$ of the rival,

$$\max\{\theta, R\} - R\mathbf{1}\{R \geq p\} = (\theta - \max\{p, R\})_+ + p\mathbf{1}\{R < p\}.$$

An additional prepared rival therefore adds $B(M_{\bar{\lambda}}) + pF(p)$ to expected allocation value at a purchase. Subtracting c_H gives $\gamma(M_{\bar{\lambda}} - \tau) + pF(p)$, and multiplying by the additional preparation probability gives the surplus formula. Both gains are positive since $M_{\bar{\lambda}} > \tau$, $\gamma > 0$, $\Delta_T > 0$, and $t_L \geq t_0$. □

Remark A.1 (Tie rules with research). Maintain the general setting. When an indifferent informed trader may mix and an indifferent high-cost rival may prepare with any probability, the outcomes above are supplemented by the following threshold choices. For $1/2 < \tau < 1$ and $0 < \lambda_E(\tau) < \bar{\lambda}$, the threshold pair has research λ_E, full high-class purchasing and the low-class sale rule of Lemma A.15. Its purchase entry is

$$e^* = \frac{2a\lambda_E + k - (\rho J - k)^+}{J},$$

and it exists exactly when $\rho \leq e^* \leq 1$. Entry at a sale and no order is ρ, and the high-cost preparation probability at a purchase is $(e^* - \rho)/(1 - \rho)$. At $e^* = 1$ it coincides with recruiting research; at $e^* = \rho$ it is floor-entry research with a purchase posterior equal to τ.

At $\lambda_E(\tau) = \bar{\lambda}$, the complete threshold family has purchase entry

$$e(1) \in \left[\max\left\{\rho, \frac{2a\bar{\lambda} + k - (\rho J - k)^+}{J}\right\}, 1\right],$$

provided the lower endpoint is at most one. Each choice has its own competitive purchase price, research at the cap, and total entry $\rho + (e(1) - \rho)[\beta + (1/2 - \beta)\bar{\lambda}]$.

When an indifferent informed trader abstains, a positive-research candidate at intensity λ, including a threshold pair, survives exactly when

$$\rho\Delta_T \leq 2k \quad \text{or} \quad \rho\Delta_T m_\lambda > k.$$

At zero research the abstention rule selects abstention after every hypothetical success. Research itself is a unique best response at fixed prices and preparation, because its objective is strictly

concave.

Proof. Positive research has the strict full purchase of Lemma A.14. Partial high-cost preparation can therefore occur only at its posterior $M_\lambda = \tau$, so $\lambda = \lambda_E(\tau)$. The sale lemma still applies, because the other posteriors are below the threshold. At an interior research choice, Lemma A.12 gives

$$2a\lambda_E = e(1)J - k + (\rho J - k)^+.$$

Solving for $e(1)$ gives e^*. Feasibility of the preparation probability is exactly $\rho \le e^* \le 1$. If the sale term is zero, the purchase earns $2a\lambda_E > 0$; if it is positive, the purchase earns at least $\rho J - k > 0$. The sale rule, competitive prices, price beliefs and truthful bids then complete the tuple as in the proof of Proposition 4(a). At the two endpoints, the research equality becomes the recruiting equation or the floor equation, respectively.

At $\lambda_E = \bar{\lambda}$, research optimality requires the left side of the same display to be at most its right side. Solving this inequality and intersecting with $[\rho, 1]$ gives the entire stated interval. Each feasible purchase entry has its own competitive purchase price; only a purchase changes entry above ρ, yielding the total-entry formula.

An indifferent informed trader who abstains permits the zero sale rate when $\rho\Delta_T \le 2k$ and a full sale when $\rho\Delta_T m_\lambda > k$. If $\rho\Delta_T/2 > k \ge \rho\Delta_T m_\lambda$, zero selling would make the sale strictly profitable, while the consistent positive rate has zero sale profit and violates the abstention rule. These cases prove the survival condition. At zero research the same rule selects abstention from the best-order sets in Lemma A.16(a), and strict concavity gives the unique research response. □

Primitives and strengths. The incumbent is uniform on $[0, r]$, the two class values are $h = 10$ and $\ell = 1$, the reserve is $p = 1/2$, the prior is $1/2$, and the costs and slot satisfy

$$\rho = 1/4, \quad c_H = 6, \quad k = 1/50, \quad \beta = 0.294655, \quad a = 1/40, \quad 1 + u = \mathrm{e}.$$

$$\bar{\lambda} = \frac{u\beta}{1 + u\beta} \approx 0.336122.$$

The path is $[r_0, r_2]$, where $r_0 = 6/5$, $r_1 = 3$ and $r_2 = 18/5$. Write $\lambda_E(r) = \lambda_E(\tau(r))$, $\lambda_R(r) = \lambda_R(\Delta_T(r))$, and $\mathfrak{r}(y) = \ell + y + \sqrt{y^2 + 2\ell y}$. Define

$$\varphi_a(r) = \Delta_T(r)[1 - \tau(r)] - k - 2a\lambda_E(r), \qquad \varphi_\rho(r) = \rho\Delta_T(r)[1 - \tau(r)] - k - a\lambda_E(r),$$
$$\varphi_a(r_R) = 0, \quad r_{\bar{\lambda}} = \mathfrak{r}((2a\bar{\lambda} + k)(1 + \mathrm{e})), \quad r_N = \mathfrak{r}(2k/\rho), \quad r_S = \mathfrak{r}(k(1 + \mathrm{e})/\rho),$$
$$\varphi_\rho(r_Q) = 0, \quad r_U = \mathfrak{r}((k + a\bar{\lambda})(1 + \mathrm{e})/\rho), \quad \tau(r_C) = M_{\bar{\lambda}} = \frac{\mathrm{e}}{1 + \mathrm{e}}.$$

The roots are the unique roots on the path of their stated equations. They satisfy

$$r_0 < r_R < r_{\bar{\lambda}} < r_N < r_S < r_Q < r_U < r_1 < r_C < r_2,$$

with values

$$\begin{aligned} &r_R \approx 1.501023, && r_{\bar{\lambda}} \approx 1.677633, && r_N \approx 1.747878, && \\ &r_S \approx 2.124148, && r_Q \approx 2.166995, && r_U \approx 2.434046, && r_C \approx 3.592659. \end{aligned}$$

The ratio $(1-\rho)k/(2a\rho)$ exceeds one. Hence the sale-profit term in (A.15) vanishes at every uncapped root in $(0,1)$, and clearing its denominator gives

$$\begin{aligned} \Lambda_r(\lambda) &= 2a(1-2\beta)\lambda^2 + \big[4a\beta + k(1-2\beta) + \beta\Delta_T(r)\big]\lambda + \beta\big[2k - \Delta_T(r)\big] \\ &= 2\big[2\beta + \lambda(1-2\beta)\big]\big[a\lambda - \tfrac{1}{2}\big(\Delta_T(r)m_\lambda - k\big)\big]. \end{aligned} \tag{A.16}$$

Every coefficient except the constant is positive, so Λ_r rises strictly on $\lambda \geq 0$. If $\Delta_T(r) > 2k$, then $\Lambda_r(0) < 0 < \Lambda_r(1)$, and $\lambda_R(r)$ is the unique root of Λ_r in $(0,1)$.

Since $m_{\lambda_E} = 1-\tau$, $\Lambda_r(\lambda_E(r)) = -[2\beta + \lambda_E(r)(1-2\beta)]\varphi_a(r)$. Thus the sign of the recruitment margin determines whether the root reaches the recruitment intensity.

Corollary 3 (Research equilibria in the example, full statement). *Take the example on $[r_0, r_2]$, when an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares.*

(a) The equilibrium outcomes are exactly as follows:

- *on $[r_0, r_R)$, zero research alone;*
- *on $[r_R, r_N]$, zero research and recruiting research;*
- *on (r_N, r_Q), recruiting research at the cap and floor-entry research;*
- *on $[r_Q, r_C]$, recruiting research at the cap alone;*
- *on $(r_C, r_2]$, floor-entry research at the cap alone.*

Recruiting research is below its cap on $[r_R, r_{\bar{\lambda}})$ and at the cap on $[r_{\bar{\lambda}}, r_C]$. Its research rises from $\lambda_E(r_R) \approx 0.193853$ to $\bar{\lambda}$, and its entry rises from 0.500846 to 0.522757, then stays constant through r_C. Class L abstains through r_N, mixes on (r_N, r_S), and sells with probability one on $[r_S, r_C]$. Floor-entry research has full orders and entry ρ throughout its existence set. At r_N, zero research has all the hypothetical-success order mixtures in Lemma A.16(a).

(b) At r_0, the unique equilibrium has zero research and entry ρ. At each $r \in \{1.55, 1.60, 1.65\}$, exactly two equilibria coexist: zero research and recruiting research with $\lambda_E(r) < \lambda_R(r) < \bar{\lambda}$. In the recruiting equilibrium class H buys and class L abstains after a success. At r_1 and r_2, the unique equilibrium has full orders and research at the cap, with entry 0.522757 at r_1 and ρ at r_2.

Proof. Take the example with $a = 1/40$, $\bar{\lambda} \approx 0.336122$ and $r \in [r_0, r_2]$. On this interval,

$$\Delta_T(r) = \frac{(r-\ell)^2}{2r}, \qquad g_L(r) = \frac{\ell^2 - p^2}{2r}, \qquad g_H(r) = h - \frac{r}{2} - \frac{p^2}{2r}.$$

Here $\Delta_T' > 0$, $g_L' < 0$, and $\tau' = (-g_L' + \tau\Delta_T')/(g_H - g_L) > 0$. Substituting $\bar{\lambda} = u\beta/(1+u\beta)$ in m_λ gives $m_{\bar{\lambda}} = 1/(2+u) = 1/(1+\mathrm{e})$ and $M_{\bar{\lambda}} = \mathrm{e}/(1+\mathrm{e})$. The margins $M_{\bar{\lambda}} - \tau(r_1) \approx 0.026059$ and $\tau(r_2) - M_{\bar{\lambda}} \approx 0.000334$ give a unique root $r_C \in (r_1, r_2)$ of $\tau(r_C) = M_{\bar{\lambda}} = \mathrm{e}/(1+\mathrm{e})$. The ratio $(1-\rho)k/(2a\rho)$ exceeds one, and therefore exceeds the cap, so the cap spread is $(2a\bar{\lambda} + k)(1+\mathrm{e})$. Thus the left endpoint is $r_{\bar{\lambda}} = \mathfrak{r}((2a\bar{\lambda} + k)(1+\mathrm{e}))$, where $\mathfrak{r}(y) = \ell + y + \sqrt{y^2 + 2\ell y}$. The positive margin $r_N - r_{\bar{\lambda}} \approx 0.070245$, together with $r_N < r_1 < r_C$, places it before the right endpoint. Recruiting research at the cap is therefore an equilibrium exactly on $[r_{\bar{\lambda}}, r_C]$, with $r_{\bar{\lambda}} \approx 1.677633$ and $r_C \approx 3.592659$.

Recruiting research below the cap.

On $[r_0, r_N]$, a sale at the prior earns at most its trading cost, and $\tau > 1/2$, so $\rho J < k$ and $\varphi_a^* = \varphi_a = J - k - 2a\lambda_E$. In the example $r_{\bar{\lambda}} < r_N < 7/4$. For $r \in [r_0, 7/4]$, differentiating the auction formulas gives

$$\Delta_T'(r) \geq 11/72, \quad 1 - \tau(r) \geq 19/55, \quad \Delta_T(r) \leq 9/56, \quad \tau'(r) \leq 16875/211288.$$

Here Δ_T', Δ_T and τ rise, while $-g_L'$ and $\gamma = g_H - g_L$ fall. The formula $\tau' = (-g_L' + \tau\Delta_T')/\gamma$ gives the upper bound

$$\tau'(r) \leq \frac{-g_L'(r_0) + \tau(7/4)\Delta_T'(7/4)}{\gamma(7/4)} < 16875/211288.$$

Also $\lambda_E'(\tau)$ rises with τ. Substitution in $\varphi_a' = \Delta_T'(1-\tau) - [\Delta_T + 2a\lambda_E'(\tau)]\tau'$ therefore gives

$$J'(r) \geq 11/72 \cdot 19/55 - 9/56 \cdot 16875/211288 > 0, \qquad \varphi_a'(r) \geq 0.033767 > 0 \quad \text{at } a = 1/40.$$

The same bound is positive for $a < a_{\mathrm{mono}} \approx 0.161713$. The endpoint values are

$$\varphi_a^*(r_0) = -1369550353/60351152400 < 0, \qquad \varphi_a^*(r_{\bar{\lambda}}) \approx 0.017446 > 0.$$

Hence recruiting research begins at the unique zero $r_R \approx 1.501023$, reaches its cap at $r_{\bar{\lambda}} \approx 1.677633$,

and continues to $r_C \approx 3.592659$. By Lemma A.10(b), J rises on the whole path. Since λ_E also rises, for $r \in [7/4, r_2]$,

$$\varphi_a(r) \geq J(7/4) - k - 2a\lambda_E(r_2) > 0,$$
$$J(7/4) = 171/3080.$$

Thus r_R is the unique zero of φ_a on $[r_0, r_2]$.

Part (a). Lemma A.16(a) gives zero research exactly through r_N. The preceding margin and cap arguments give recruiting research on $[r_R, r_C]$, below the cap until $r_{\bar{\lambda}}$. Lemma A.18 gives the two floor-entry intervals and uniqueness above r_Q. Together with the exhaustion in Lemma A.16, these give the five intervals in the statement. The research root equals $\lambda_E(r_R)$ at the start, and the entry formula of Proposition 4(b) gives its starting and cap values. At the cap the sale lemma gives abstention through r_N, mixing up to r_S, and full selling from r_S onward.

Part (b), the weak incumbent. At r_0, the gross return of any nonzero order is at most $\Delta_T(r_0) = 1/60 < k$, with margin 0.003333. Every such order loses, so $\Psi = 0$ and research is zero. At the prior, $c_H - B_{r_0}(1/2) \approx 1.195833 > 0$, giving entry ρ.

Part (b), the intermediate strengths. Clearing the research equation gives (A.16). Its positive root has the closed form

$$\lambda_R(r) = \frac{2\beta[\Delta_T(r) - 2k]}{\Lambda_r'(0) + \sqrt{[\Lambda_r'(0)]^2 + 8a\beta(1-2\beta)[\Delta_T(r) - 2k]}}.$$

At $r \in \{1.55, 1.60, 1.65\}$, exact rational arithmetic and bounds on the square root give $1/2 < \tau(r) < 1$, with $k - \rho\Delta_T(r)/2$ equal to 0.007802, 0.005938 and 0.003996, respectively, to the displayed precision. The differences $\lambda_R(r) - \lambda_E(r)$ are approximately 0.041160, 0.079340 and 0.114019; the differences $\bar{\lambda} - \lambda_R(r)$ are approximately 0.098221, 0.057085 and 0.019439. All are strictly positive. Lemma A.16 therefore gives exactly zero research and recruiting research below the cap; the sale lemma gives strict low-class abstention. The entry formula gives the values in Table 4.

Part (b), the strong and the very strong incumbent. The cap posterior lies strictly between $\tau(r_1)$ and $\tau(r_2)$, with margins 0.026059 and 0.000334. Condition (U) holds at both strengths, with margins 0.016421 and 0.034723. Lemma A.17 gives uniqueness, full orders and research at the cap, with entry 0.522757 at r_1 and ρ at r_2. Class H outbids the incumbent throughout the path because $r \leq r_2 < h$, so its ownership is $\frac{1}{2}[\rho + (1-\rho)(\beta + (1-\beta)\hat{\lambda})]$. At r_1, the price-access formulas give the proceeds gain 0.257948 and the surplus gain 0.081960. □

Floor-entry research. For $\rho\Delta_T(r) > 2k$, with λ_Q the root of (A.11), *floor-entry research* is the tuple with $\hat{\lambda} = \min\{\lambda_Q(\Delta_T(r)), \bar{\lambda}\}$, $\sigma_H = \delta_1$, $\sigma_L = \delta_{-1}$, $(\mu(-1), \mu(0), \mu(1)) = (m_{\hat{\lambda}}, 1/2, M_{\hat{\lambda}})$,

$\zeta \equiv 0$, prices $P(q) = t_0 + \rho\, w(\mu(q))$, $\bar{\mu}(P(q)) = \mu(q)$ and truthful bids. Its entry is ρ.

Recruiting research at the cap is the tuple with $\hat{\lambda} = \bar{\lambda}$, $\sigma_H = \delta_1$, $\sigma_L(-1) = \hat{s}_-(r)/\bar{\lambda}$ and $\sigma_L(0) = 1 - \hat{s}_-(r)/\bar{\lambda}$, where $\hat{s}_-(r)$ is the sale rate of Lemma A.15 at the presence $\bar{\lambda}$:

$$\hat{s}_-(r) = \min\{\bar{\lambda},\ \bar{b}\,[\rho\Delta_T(r)/k - 2]^+\}, \qquad \bar{b} = (1-\bar{\lambda})\beta, \qquad \bar{b}_0 = (1-\bar{\lambda})(1-2\beta).$$

Its posteriors are $\mu(-1) = \bar{b}/\big(2\bar{b} + \hat{s}_-(r)\big)$, $\mu(0) = \bar{b}_0/\big(2\bar{b}_0 + \bar{\lambda} - \hat{s}_-(r)\big)$ and $\mu(1) = M_{\bar{\lambda}}$. It has $\zeta(P(1)) = 1$ and $\zeta(P(0)) = \zeta(P(-1)) = 0$, prices $P(q) = t_0 + e(q)\, w(\mu(q))$, $\bar{\mu}(P(q)) = \mu(q)$ and truthful bids. Class L abstains when $\rho\Delta_T \le 2k$ and sells with probability one when $\rho\Delta_T m_{\bar{\lambda}} \ge k$; in between it mixes, and its sale just pays the trading cost.

The *floor margin* is

$$\varphi_\rho(r) = \rho\Delta_T(r)\,[1 - \tau(r)] - k - a\lambda_E(r).$$

Since $m_{\lambda_E(r)} = 1 - \tau(r)$, it is the gap $\rho\Delta_T m_\lambda - k - a\lambda$ of (A.11) at $\lambda = \lambda_E(r)$: the return to a full order at the entry floor, at the research that lets a purchase just reach the threshold, net of the trading cost and of the marginal cost of that research.

Lemma A.18 (Floor-entry research along the path)**.** *Take the example on $[r_0, r_2]$, when an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares.*

(a) The floor margin φ_ρ rises strictly on the path. Its unique zero is r_Q, with $r_N < r_Q < r_U < r_C$. On $(r_N, r_C]$, the threshold exceeds the top posterior at research $\min\{\lambda_Q(\Delta_T(r)), \bar{\lambda}\}$ exactly on (r_N, r_Q), and equals it exactly at r_Q and r_C.

(b) Floor-entry research is an equilibrium exactly on (r_N, r_Q) and $(r_C, r_2]$, with entry ρ. The floor root reaches the cap at r_U. On $[r_N, r_Q]$, it rises strictly from zero to $\lambda_E(r_Q) \approx 0.234257 < \bar{\lambda}$; on $(r_C, r_2]$ research is at the cap.

(c) Recruiting research at the cap is an equilibrium exactly on $[r_{\bar{\lambda}}, r_C]$, with entry 0.522757.

(d) On (r_N, r_Q), exactly these two equilibria coexist. On $[r_Q, r_C]$, recruiting research at the cap is unique; on $(r_C, r_2]$, floor-entry research at the cap is unique.

Proof. Throughout, $r \in [r_0, r_2]$, λ_Q stands for $\lambda_Q(\Delta_T(r))$, and *the gap* is $\rho\Delta_T(r)\, m_\lambda - k - a\lambda$, the gap of (A.11) at $\Delta_T = \Delta_T(r)$, as a function of λ. Recall $r_N = \mathfrak{r}(2k/\rho)$, $r_{\bar{\lambda}} = \mathfrak{r}\big((2a\bar{\lambda} + k)(1+\mathrm{e})\big)$, $r_U = \mathfrak{r}\big((k + a\bar{\lambda})(1+\mathrm{e})/\rho\big)$, and the entry ceiling r_C, at which $\tau(r_C) = M_{\bar{\lambda}}$.

Step 0: facts along the path. The spread Δ_T and the threshold τ rise strictly. Since $B_r(1/2) = h/2 - r/4 + (\ell^2 - 2p^2)/(4r) < c_H$, $\tau(r) > 1/2$, and $\tau(r) \le \tau(r_2) = 4245/5804 < 1$; so $\lambda_E(r) \in (0,1)$ is defined and rises strictly. Since M_λ rises strictly in λ and $M_{\lambda_E(r)} = \tau(r)$,

$\lambda_E(r) \le \bar\lambda$ exactly when $r \le r_C$, with equality only at r_C. Condition (U) holds at r_1, with margin 0.016421, and $M_{\bar\lambda} - \tau(r_1) \approx 0.026059$ and $\tau(r_2) - M_{\bar\lambda} \approx 0.000334$, so $r_U < r_1 < r_C < r_2$. Also $r_N - r_{\bar\lambda} \approx 0.070245 > 0$. Since $m_{\bar\lambda} = 1/(1+\mathrm{e})$, the gap at $\bar\lambda$, $\rho\Delta_T(r)\, m_{\bar\lambda} - k - a\bar\lambda$, is nonnegative exactly when $r \ge r_U$, with equality only at r_U. For $r \ge r_N$, the gap falls strictly in λ and vanishes at λ_Q, so $\lambda_Q \ge \bar\lambda$ exactly when $r \ge r_U$.

Step 1: the comparison. Let $r \in (r_N, r_C]$. Then $\lambda_E(r) \le \bar\lambda$, and $m_{\lambda_E(r)} = 1 - \tau(r)$ makes the gap at $\lambda_E(r)$ equal to $\varphi_\rho(r)$. Since M_λ rises strictly, $\tau(r)$ exceeds M_λ at $\lambda = \min\{\lambda_Q, \bar\lambda\}$ exactly when $\min\{\lambda_Q, \bar\lambda\} < \lambda_E(r)$. Because $\lambda_E(r) \le \bar\lambda$, this is $\lambda_Q < \lambda_E(r)$, and because the gap falls strictly in λ and vanishes at λ_Q, it is $\varphi_\rho(r) < 0$. In the same way, $\tau(r)$ equals M_λ at $\lambda = \min\{\lambda_Q, \bar\lambda\}$ exactly when $\min\{\lambda_Q, \bar\lambda\} = \lambda_E(r)$, which holds exactly when $\lambda_Q = \lambda_E(r)$, that is $\varphi_\rho(r) = 0$, or when $\bar\lambda = \lambda_E(r) \le \lambda_Q$, that is $r = r_C$, by Step 0 and $r_C > r_U$.

Step 2: the floor margin rises. By the formula for $d\lambda_E/d\tau$,

$$\varphi_\rho'(r) = \rho\big[\Delta_T'(r)\,(1-\tau(r)) - \Delta_T(r)\,\tau'(r)\big] - \frac{a\beta\,\tau'(r)}{\{\beta + (1-2\beta)[1-\tau(r)]\}^2},$$

with $\Delta_T'(r) = \frac{1}{2}(1 - \ell^2/r^2)$ and $\tau'(r) = 2[c_H r^2 - (\ell^2 - p^2)r + h(\ell^2 - p^2) - c_H\ell^2]/(2hr - r^2 - \ell^2)^2$. On $[r_0, r_2]$ the factors Δ_T', Δ_T, $1 - \tau$ and τ' are positive. On a piece $[r', r''] \subset [r_0, r_2]$, Δ_T', Δ_T and τ rise, $c_H r^2 - (\ell^2 - p^2)r + h(\ell^2 - p^2) - c_H\ell^2$ rises, and $\tau' \le T(r', r'')$, since $2hr - r^2 - \ell^2$ is positive and rises; the denominator $\{\beta + (1-2\beta)[1-\tau]\}^2$ of the last term falls as τ rises, so on the piece it is at least its value at r''. Hence, for $r \in [r', r'']$,

$$\varphi_\rho'(r) \ge \rho\big[\Delta_T'(r')\,(1-\tau(r'')) - \Delta_T(r'')\,T(r', r'')\big] - \frac{a\beta\,T(r', r'')}{\{\beta + (1-2\beta)[1-\tau(r'')]\}^2},$$
$$T(r', r'') = \frac{2[c_H(r'')^2 - (\ell^2 - p^2)r'' + h(\ell^2 - p^2) - c_H\ell^2]}{(2hr' - (r')^2 - \ell^2)^2}.$$

Partition $[r_0, r_2]$ into pieces of width $1/10$. An exact rational computation shows that the right side of the first line is at least $1/100$ on each piece, so φ_ρ rises strictly on $[r_0, r_2]$.

Step 3: the zero. At r_N, $\rho\Delta_T(r_N) = 2k$, so $\varphi_\rho(r_N) = k[1 - 2\tau(r_N)] - a\lambda_E(r_N) < 0$. Since $m_{\bar\lambda} = 1 - M_{\bar\lambda}$,

$$\varphi_\rho(r) = \big[\rho\Delta_T(r)\, m_{\bar\lambda} - k - a\bar\lambda\big] + \rho\Delta_T(r)\big[M_{\bar\lambda} - \tau(r)\big] + a\big[\bar\lambda - \lambda_E(r)\big].$$

On $[r_U, r_C]$, the first bracket is nonnegative, with equality only at r_U, and the other two terms are nonnegative and vanish only at r_C. Since $r_U < r_C$, $\varphi_\rho > 0$ on $[r_U, r_C]$. By Step 2, φ_ρ has exactly one zero r_Q on $[r_0, r_2]$, with $r_N < r_Q < r_U$; it is negative on $[r_0, r_Q)$ and positive on $(r_Q, r_2]$. The floor margin is a rational function of r with rational coefficients, so its sign at a rational strength follows from exact rational arithmetic; these signs give $r_Q \approx 2.166995$ and $\lambda_E(r_Q) \approx 0.234257$.

With Step 1, this proves part (a).

Step 4: floor-entry research. Let $r \in (r_N, r_2]$, so $\rho\Delta_T(r) > 2k$, and take floor-entry research, with $\hat{\lambda} = \min\{\lambda_Q, \bar{\lambda}\} > 0$. (E1): $f_H = (\hat{b}, \hat{b}_0, \hat{b} + \hat{\lambda})$ and $f_L = (\hat{b} + \hat{\lambda}, \hat{b}_0, \hat{b})$ on $(-1, 0, 1)$ give the stated posteriors, and $m_{\hat{\lambda}} < 1/2 < M_{\hat{\lambda}}$. (E2): the stated entry is ρ after every order, so by the paragraph on constructed tuples in Appendix A.2 the three prices differ, and each price reveals its order. Suppose $\tau(r) > M_{\hat{\lambda}}$. (E3): every posterior is at most $M_{\hat{\lambda}} < \tau(r)$, so $B_r < c_H$ at every price, and $\zeta \equiv 0$ with no tie. (E4): (A.4). (E5): by (A.12), $U_H(1) = \rho\Delta_T(1 - M_{\hat{\lambda}}) - k$ and $U_L(-1) = \rho\Delta_T m_{\hat{\lambda}} - k$. Both equal $\rho\Delta_T m_{\hat{\lambda}} - k$, which is at least $a\hat{\lambda} > 0$, because $\hat{\lambda} \le \lambda_Q$ and the gap is nonnegative on $[0, \lambda_Q]$. With $U_j(0) = 0$ and Lemma A.13(c), the purchase and the sale are the strict best orders. (E6): truthful bids. (E7): $\Psi = \rho\Delta_T m_{\hat{\lambda}} - k$. If $\lambda_Q < \bar{\lambda}$, then $\hat{\lambda} = \lambda_Q$ and $\Psi/a = \lambda_Q < \bar{\lambda}$; if $\lambda_Q \ge \bar{\lambda}$, then $\hat{\lambda} = \bar{\lambda}$ and $\Psi - a\bar{\lambda} = \rho\Delta_T m_{\bar{\lambda}} - k - a\bar{\lambda} \ge 0$. In both cases $\min\{\bar{\lambda}, \Psi/a\} = \hat{\lambda}$. Conversely, if $\tau(r) \le M_{\hat{\lambda}}$, then $B_r(\mu(1)) \ge c_H$, and (E3) with an indifferent rival that prepares requires $\zeta(P(1)) = 1$, so the tuple is not an equilibrium.

Floor-entry research is therefore an equilibrium exactly when $\tau(r) > M_{\hat{\lambda}}$ at $\hat{\lambda} = \min\{\lambda_Q, \bar{\lambda}\}$. By part (a), on $(r_N, r_C]$ this holds exactly on (r_N, r_Q). On $(r_C, r_2]$, $r > r_U$ gives $\min\{\lambda_Q, \bar{\lambda}\} = \bar{\lambda}$, and $\tau(r) > M_{\bar{\lambda}}$, so it holds there, with research at the cap. On $[r_N, r_Q]$, $r < r_U$, so $\lambda_Q < \bar{\lambda}$; it rises strictly and continuously with r, from 0 at r_N, where the gap at $\lambda = 0$ is $\rho\Delta_T(r_N)/2 - k = 0$, to $\lambda_E(r_Q)$ at r_Q, where the gap at $\lambda_E(r_Q)$ is $\varphi_\rho(r_Q) = 0$. This proves part (b).

Step 5: recruiting research at the cap. Let $r \in [r_{\bar{\lambda}}, r_C]$, and write $\hat{s}_- = \hat{s}_-(r)$. (E1): $f_H = (\bar{b}, \bar{b}_0, \bar{b} + \bar{\lambda})$ and $f_L = (\bar{b} + \hat{s}_-, \bar{b}_0 + \bar{\lambda} - \hat{s}_-, \bar{b})$ give the stated posteriors, with $\mu(-1), \mu(0) \le 1/2 < \tau(r) \le M_{\bar{\lambda}} = \mu(1)$. (E2): the stated entry, ρ after a sale and after no order and 1 after a purchase, is nondecreasing in the posterior; by the paragraph on constructed tuples, orders with equal posteriors have equal entry ρ and share a price, orders with distinct posteriors have distinct prices, and the belief at each price is the posterior of its orders. (E3): $B_r(M_{\bar{\lambda}}) \ge c_H$, with equality only at r_C, where the indifferent rival prepares, and $B_r < c_H$ at the posteriors at most $1/2$. (E4): (A.4). (E5): the purchase earns $\Delta_T m_{\bar{\lambda}} - k \ge \Delta_T(r_{\bar{\lambda}})\, m_{\bar{\lambda}} - k = 2a\bar{\lambda} > 0$, so by Lemma A.13(c) it is the strict best order of class H. Class L earns $U_L(1) \le -k$, $U_L(0) = 0$ and $U_L(-1) = \rho\Delta_T\mu(-1) - k$. If $\hat{s}_- = 0$, then $\rho\Delta_T \le 2k$ and $U_L(-1) = \rho\Delta_T/2 - k \le 0$. If $0 < \hat{s}_- < \bar{\lambda}$, then $2\bar{b} + \hat{s}_- = \bar{b}\rho\Delta_T/k$, and $U_L(-1) = 0$. If $\hat{s}_- = \bar{\lambda}$, then $\rho\Delta_T m_{\bar{\lambda}} \ge k$, since $m_{\bar{\lambda}} = \bar{b}/(2\bar{b} + \bar{\lambda})$, and $U_L(-1) = \rho\Delta_T m_{\bar{\lambda}} - k \ge 0$. In each case σ_L puts weight only on best orders, and an indifferent informed trader may mix. (E6): truthful bids. (E7): $\Psi \ge \frac{1}{2}(\Delta_T m_{\bar{\lambda}} - k) \ge a\bar{\lambda}$, so $\min\{\bar{\lambda}, \Psi/a\} = \bar{\lambda}$. A purchase has probability $\frac{1}{2}(\bar{b} + \bar{\lambda}) + \frac{1}{2}\bar{b} = \beta + (1/2 - \beta)\bar{\lambda}$, and only the purchase raises entry above ρ, so entry is $\rho + (1 - \rho)[\beta + (1/2 - \beta)\bar{\lambda}] \approx 0.522757$.

For $r > r_C$, $\mu(1) = M_{\bar{\lambda}} < \tau(r)$, so (E3) requires $\zeta(P(1)) = 0$, and the tuple is not an equilibrium. For $r < r_{\bar{\lambda}}$, $r < r_N$ gives $\hat{s}_- = 0$ and $U_L(-1) < 0$, so $\Psi = \frac{1}{2}(\Delta_T m_{\bar{\lambda}} - k)^+ < a\bar{\lambda}$, because $\Delta_T(r) < \Delta_T(r_{\bar{\lambda}})$; (E7) fails. This proves part (c).

Step 6: every equilibrium above r_N. Let $r \in (r_N, r_C]$, and take any equilibrium. If $\hat{\lambda} = 0$, every posterior is $1/2$, entry is ρ after every order, and $U_H(1) = \rho\Delta_T/2 - k > 0$; then $\Psi > 0$, and Lemma A.12 gives $\hat{\lambda} > 0$, a contradiction. So $\hat{\lambda} > 0$, and Lemma A.14 gives $\sigma_H = \delta_1$, $\mu(1) = M_{\hat{\lambda}}$ and $e(0) = e(-1) = \rho$. Lemma A.2(d) gives $e(1) = 1$ if $M_{\hat{\lambda}} \geq \tau(r)$ and $e(1) = \rho$ otherwise.

If $M_{\hat{\lambda}} < \tau(r)$, then $U_H(1) = \rho\Delta_T m_{\hat{\lambda}} - k$, which is positive by Lemma A.14. Lemma A.13(a) gives $\mu(-1) \geq m_{\hat{\lambda}}$, so $U_L(-1) \geq \rho\Delta_T m_{\hat{\lambda}} - k > 0$, and the sale is the unique best order of class L. Then $\sigma_L = \delta_{-1}$, $\mu(-1) = m_{\hat{\lambda}}$ and $\Psi = \rho\Delta_T m_{\hat{\lambda}} - k$. Lemma A.12 gives $\hat{\lambda} = \min\{\bar{\lambda}, \Psi/a\}$: if $\hat{\lambda} < \bar{\lambda}$, then $a\hat{\lambda} = \rho\Delta_T m_{\hat{\lambda}} - k$, so the gap vanishes at $\hat{\lambda}$ and $\hat{\lambda} = \lambda_Q < \bar{\lambda}$; if $\hat{\lambda} = \bar{\lambda}$, then the gap at $\bar{\lambda}$ is nonnegative, and $\lambda_Q \geq \bar{\lambda}$. So $\hat{\lambda} = \min\{\lambda_Q, \bar{\lambda}\}$. The orders give the posteriors, Lemma A.2 gives the preparation, the prices and the beliefs, and the bids are truthful, so the tuple is floor-entry research.

If $M_{\hat{\lambda}} \geq \tau(r)$, then $e(1) = 1$ and $U_H(1) = \Delta_T m_{\hat{\lambda}} - k$. Class L can abstain, so $\Psi \geq \frac{1}{2}(\Delta_T m_{\hat{\lambda}} - k)$. The map $\lambda \mapsto \Delta_T(r)\, m_\lambda - k - 2a\lambda$ falls strictly and is nonnegative at $\bar{\lambda}$, because $r \geq r_{\bar{\lambda}}$. If $\hat{\lambda} < \bar{\lambda}$, then $a\hat{\lambda} = \Psi \geq \frac{1}{2}(\Delta_T m_{\hat{\lambda}} - k) > a\hat{\lambda}$, a contradiction; so $\hat{\lambda} = \bar{\lambda}$. Let $s_- = \bar{\lambda}\,\sigma_L(-1)$ be the low-class sale rate. Lemma A.13(c) gives $\sigma_L(1) = 0$, so $\mu(-1) = \bar{b}/(2\bar{b} + s_-)$, and $U_L(-1) = \rho\Delta_T \bar{b}/(2\bar{b} + s_-) - k$ falls strictly in s_-. If $\rho\Delta_T m_{\bar{\lambda}} \geq k$, then $U_L(-1) > 0$ at every $s_- < \bar{\lambda}$, and (E5) gives $s_- = \bar{\lambda}$. Otherwise $U_L(-1)$ is positive at $s_- = 0$, because $\rho\Delta_T > 2k$, and negative at $s_- = \bar{\lambda}$, so (E5) admits only the $s_- \in (0, \bar{\lambda})$ at which $U_L(-1) = 0$. So $s_- = \hat{s}_-(r)$, and as in the first case the tuple is recruiting research at the cap.

Every equilibrium on $(r_N, r_C]$ is therefore floor-entry research or recruiting research at the cap. By parts (b) and (c), the equilibria are exactly these two on (r_N, r_Q), and recruiting research at the cap alone on $[r_Q, r_C]$. For $r \in (r_C, r_2]$, $r > r_U$ gives (U), and $M_{\bar{\lambda}} < \tau(r)$. Lemma A.17 then gives a unique equilibrium, with research at the cap, full orders and $\zeta(P(1)) = 0$, which is floor-entry research at the cap. This proves part (d). □

The research $\lambda_Q(\Delta_T(r))$ of floor-entry research and the recruitment intensity $\lambda_E(r)$ both rise with strength. On $(r_N, r_C]$, the floor margin has the sign of $\lambda_Q(\Delta_T(r)) - \lambda_E(r)$; it is negative at r_N, where λ_Q starts from zero, and it rises, so λ_Q crosses the recruitment intensity once, at r_Q. From r_Q to r_C, a purchase at the research $\min\{\lambda_Q, \bar{\lambda}\}$ recruits, and only recruiting research at the cap remains. Above r_C, even research at the cap leaves the purchase posterior below the threshold, and entry returns to the floor.

Proposition A.1 (The research cap and cost)**.** *Keep the example's primitives other than a and $\bar{\lambda}$, with $\beta = 0.294655$. Under an indifferent informed trader who may mix and an indifferent high-cost rival who prepares, the six-strength equilibrium pattern in Corollary 3(b), with $\lambda_E(r) < \lambda_R(r) < \bar{\lambda}$ at each intermediate strength, holds with strict margins exactly when*

$$\lambda_E(r_1) < \bar{\lambda} < \lambda_E(r_2), \qquad a_{\mathrm{lo}}(\bar{\lambda}) < a < a_{\mathrm{up}},$$

where

$$a_{\mathrm{lo}}(\bar\lambda) = \frac{\Delta_T(1.65)m_{\bar\lambda} - k}{2\bar\lambda}, \qquad a_{\mathrm{up}} = \frac{\Delta_T(1.55)[1-\tau(1.55)] - k}{2\lambda_E(1.55)}.$$

The cap strip is approximately $(0.290539, 0.336721)$. *At the example's cap,* $\bar\lambda \approx 0.336122$*, the cost window is approximately* $(0.021469, 0.036763)$*; at* $a = 1/40$ *the cap range is approximately* $(0.316683, 0.336721)$. *Entry at* r_1 *is* $\rho + (1-\rho)[\beta + (1/2-\beta)\bar\lambda]$. *With* β *varying,* $1 + u = \mathrm{e}$*,* $\bar\lambda = \bar\lambda(\beta) = u\beta/(1+u\beta)$ *and* $a = 1/40$*, the six-strength pattern, with entry* $\rho+(1-\rho)[\beta+(1/2-\beta)\bar\lambda]$ *at* r_1*, holds for* $\beta \in (\beta_{\mathrm{lo}}, 1/3]$*, where* $\beta_{\mathrm{lo}} \approx 0.236157$.

Proof of Proposition A.1. Fix $\beta = 0.294655$ and the other primitives of the example. The required equilibrium patterns are those of Corollary 3(b) with $\lambda_E(r) < \lambda_R(r) < \bar\lambda$ at each intermediate strength.

Sufficiency. Let the two conditions hold. The weak-incumbent claim holds for every $(\bar\lambda, a)$, since $\Delta_T(r_0) < k$. At the intermediate strengths, $1/2 < \tau < 1$ and $\rho\Delta_T < 2k$ do not involve $(\bar\lambda, a)$ and hold at the three strengths, so Lemma A.16 leaves zero research and recruiting research. The condition $a < a_{\mathrm{up}}$ is $\varphi_a(1.55) > 0$. Since $a_{\mathrm{up}} < a_{\mathrm{mono}}$, φ_a rises on $[r_0, 7/4]$, so $\varphi_a > 0$ at 1.60 and 1.65 too. The condition $a > a_{\mathrm{lo}}(\bar\lambda)$ is $\Lambda_{1.65}(\bar\lambda) > 0$, that is $\lambda_R(1.65) < \bar\lambda$. Since λ_R rises in r, $\lambda_R < \bar\lambda$ at the two lower strengths too. Then $\lambda_E < \lambda_R < \bar\lambda$, and Lemma A.16 gives the sets. At the strong and the very strong incumbent, M_λ rises in λ and $M_{\lambda_E(r)} = \tau(r)$, so the strip gives $\tau(r_1) < M_{\bar\lambda} < \tau(r_2)$. Condition (U) at r_1 is $a < a_U(r_1; \bar\lambda) := [\rho\Delta_T(r_1)m_{\bar\lambda} - k]/\bar\lambda$, and at r_2 it is $a < a_U(r_2; \bar\lambda) := [\rho\Delta_T(r_2)m_{\bar\lambda} - k]/\bar\lambda$. Both bounds fall in $\bar\lambda$, so on the strip they exceed their values at $\bar\lambda = \lambda_E(r_2)$, which are 0.073556 and 0.127845. Both exceed $a_{\mathrm{up}} = 0.036763$, so Lemma A.17 applies at both strengths. Its entry formula gives $\rho + (1-\rho)[\beta + (1/2-\beta)\bar\lambda]$ at r_1.

Necessity. Each failure of a condition breaks one claim.

- $\bar\lambda \le \lambda_E(r_1)$: every posterior at r_1 is at most $M_{\bar\lambda} \le \tau(r_1)$, so entry above ρ needs a tie for the rival, and strict recruitment at r_1 fails.
- $\bar\lambda \ge \lambda_E(r_2)$: research at the cap gives $\mu(1) = M_{\bar\lambda} \ge \tau(r_2)$, so an indifferent high-cost rival who prepares raises entry above ρ at r_2, and exclusion at r_2 fails.
- $a \ge a_{\mathrm{up}}$: $\varphi_a(1.55) \le 0$, so recruiting research below the cap is absent at 1.55, or has $\mu(1) = \tau$ with a tie for the rival.
- $a \le a_{\mathrm{lo}}(\bar\lambda)$: $\Lambda_{1.65}(\bar\lambda) \le 0$, so research at 1.65 sits at the cap.

The cost window is nonempty on the whole strip: a_{lo} falls in $\bar\lambda$, because its numerator falls and stays positive there, and $a_{\mathrm{lo}}(\lambda_E(r_1)) = 0.030579 < a_{\mathrm{up}}$. The cap strip is $(0.290539, 0.336721)$; the margins of $\bar\lambda \approx 0.336122$ are 0.045583 and 0.000599, and the margins of $a = 1/40$ in its

window are 0.003531 and 0.011763. The upper cap margin corresponds to the posterior margin $\tau(r_2) - M_{\bar\lambda} \approx 0.000334$; the two numbers measure one boundary in two coordinates.

General β. Set $\bar\lambda = \bar\lambda(\beta) = u\beta/(1+u\beta)$ with $1+u = \mathrm{e}$, and keep $a = 1/40$. Then $m_{\bar\lambda} = 1/(2+u) = 1/(1+\mathrm{e})$ for every β, so $M_{\bar\lambda} = \mathrm{e}/(1+\mathrm{e})$ and $m_{\bar\lambda}$ do not depend on β, and the endpoint conditions $\tau(r_1) < M_{\bar\lambda} < \tau(r_2)$ hold for every β. Condition (U) at r_1 holds for every $\beta \le 1/3$, because $\bar\lambda(\beta)$ rises in β and $\rho\Delta_T(r_1)/(1+\mathrm{e}) - k - a\bar\lambda(1/3) > 0$; it also holds at r_2, since $\Delta_T(r_2) > \Delta_T(r_1)$. At the intermediate strengths, $a_{\mathrm{lo}} = [\Delta_T(1.65)/(1+\mathrm{e}) - k]/[2\bar\lambda(\beta)]$ falls in β, so $a_{\mathrm{lo}} < a$ exactly when $\beta > \beta_{\mathrm{lo}} \approx 0.236157$. Here β_{lo} is the unique solution of

$$2a\frac{u\beta}{1+u\beta} = \frac{\Delta_T(1.65)}{1+\mathrm{e}} - k;$$

the left side rises strictly from zero, so the equality determines the boundary. The upper bound $a_{\mathrm{up}}(\beta)$ falls in β, because $\partial\lambda_E/\partial\beta = (2\tau-1)(1-\tau)/\{\beta + (1-2\beta)(1-\tau)\}^2 > 0$, and it exceeds a for every $\beta < 0.4897$. The monotonicity bound $a_{\mathrm{mono}}(\beta)$ falls in β on $(0, 1/3]$ and equals $0.150885 > a$ at $\beta = 1/3$. The steps above then apply at each $\beta \in (0.236157, 1/3]$.

□

Remark A.1, the example. *At the three intermediate strengths, allowing an indifferent high-cost rival to prepare with any probability adds the threshold pair, whose entry is* 0.456294*,* 0.420777 *and* 0.392494*, respectively. At r_C, its cap family permits every $e(1) \in [\rho, 1]$, and total entry fills* $[\rho, 0.522757]$.

When an indifferent informed trader abstains and an indifferent high-cost rival prepares, recruiting research is absent on $(r_N, r_S]$, and floor-entry research is the unique equilibrium there. When an indifferent high-cost rival may prepare with any probability, the threshold pair at $r = 2$ has

$$\begin{aligned} \lambda = \lambda_E(r) &\approx 0.223835, \\ e(1) &\approx 0.359944, \\ \mathsf{E} &\approx 0.287449. \end{aligned}$$

Both trades are strictly profitable in this pair, so it is also an equilibrium when an indifferent informed trader abstains. At all strengths above r_U other than r_C, the equilibrium is unique under all four tie rules.

Proof. At each intermediate strength the sale payoff is strictly negative. Thus $e^* = (2a\lambda_E + k)/J$, with the respective values 0.865703, 0.758777 and 0.673748. Substitution in $\rho + (e^* - \rho)[\beta + (1/2 - \beta)\lambda_E]$ gives the three entry values. At $r_C > r_U$, condition (U) makes cap research optimal even at purchase entry ρ; the cap family in Remark A.1 therefore has the whole interval $[\rho, 1]$.

On the recruiting segment at the cap, $\rho\Delta_T > 2k$ and $\rho\Delta_T m_{\bar\lambda} \le k$ hold together exactly on

$(r_N, r_S]$. The survival condition in Remark A.1 excludes recruiting research there when an indifferent informed trader abstains. This interval lies inside (r_N, r_Q), where floor-entry research has strict orders and a strict preparation decision, so that outcome remains unique with full preparation at a tie for the rival. At $r = 2$, substituting the rational auction quantities into λ_E, e^*, and the total-entry formula gives the displayed values. There $\rho J - k > 0$ and $e^* > \rho$, so both trades are strictly profitable and the threshold pair survives the rule that an indifferent informed trader abstains. Finally, every $r > r_U$ satisfies condition (U); except at r_C, its cap posterior differs from the threshold, and Lemma A.17 gives uniqueness. □

A.6 Robustness

This subsection states and proves Proposition A.2 and the results that Section 6 summarizes. Each proof uses the conventions of Appendix A.2.

The fixed-information control

Proposition A.2 (Fixed information)**.** *Fix the joint distribution of the class j and a signal S that the rival observes. Fix the class value distributions G_j and the cost distribution. In each economy i, require that, conditional on (j, S), the joint distribution of (θ, R) is $G_j \otimes F_i$. The cost is independent of the class, both values and S. If the incumbent strengthens from F_0 to F_1, entry is nonincreasing when an indifferent high-cost rival prepares. Its decrease is exactly*

$$\mathsf{E}_0 - \mathsf{E}_1 = \Pr\big\{B_{F_1}(\mu(S)) < C \le B_{F_0}(\mu(S))\big\}, \qquad \mu(S) = \Pr(j = H \mid S),$$

and the decrease is strict exactly when this probability is positive.

Proof of Proposition A.2. The cost is independent of the class, the values and the signal S, so $\mu(S) = \Pr(j = H \mid S)$ is the rival's complete class belief at its decision. The conditional joint distribution $G_j \otimes F_i$ gives gross profit $B_{F_i}(\mu(S))$ in economy i. When an indifferent high-cost rival prepares, the rival prepares exactly when $C \le B_{F_i}(\mu(S))$. Proposition 1(a) gives $B_{F_1}(\mu) \le B_{F_0}(\mu)$ at every belief. Hence, pointwise,

$$\mathbf{1}\{C \le B_{F_0}(\mu(S))\} - \mathbf{1}\{C \le B_{F_1}(\mu(S))\} = \mathbf{1}\{B_{F_1}(\mu(S)) < C \le B_{F_0}(\mu(S))\} \ge 0.$$

The joint distribution of (S, C) is the same in both economies. Take expectations: $\mathsf{E}_0 - \mathsf{E}_1$ equals the probability in the proposition, which is nonnegative, and it is positive exactly when the decrease is strict. A tie probability q that is the same in both economies assigns the preparation probability $\mathbf{1}\{C < B_{F_i}\} + q\,\mathbf{1}\{C = B_{F_i}\}$. With $B_{F_1} \le B_{F_0}$, the difference of the two expressions is 0 for $C < B_{F_1}$

and for $C > B_{F_0}$, $1-q$ at $C = B_{F_1} < B_{F_0}$, 1 strictly between, q at $C = B_{F_0} > B_{F_1}$, and 0 when the two profits are equal; it is nonnegative, so entry is nonincreasing under that common tie probability. □

The control in the slot. The experiment fixes λ, β, π, the cost distribution and the full orders $(\sigma_H, \sigma_L) = (\delta_1, \delta_{-1})$. Lemma A.1(d) gives the posteriors (m, π, M). Let $e_r(\mu) = \rho + (1-\rho)\mathbf{1}\{B_r(\mu) \geq c_H\}$ and $P(x) = t_0 + e_r(\mu(x))\, w(\mu(x))$. This entry is nondecreasing in the posterior, so by the paragraph on constructed tuples in Appendix A.2 the three prices differ, and each price reveals its posterior. The rival's signal is therefore the order, with a distribution that does not depend on r. Given the class and order, the value distribution is $G_j \otimes F_r$, as Proposition A.2 requires. In the example, on $\ell \leq r \leq h$, $\tau(r) = [2rc_H - (\ell^2 - p^2)]/(2rh - r^2 - \ell^2)$ rises and

$$\tau(\ell) = \frac{2\ell c_H - (\ell^2 - p^2)}{2\ell(h-\ell)} > \pi,$$

so only the top posterior M can recruit, and it recruits exactly when $r \leq r_C$. Entry is therefore $\rho + (1-\rho)(b + \lambda/2) = 0.522757$ on $(\ell, r_C]$ and $\rho = 0.250000$ above $r_C = 3.592659$. The probability of Proposition A.2 is 0 from r_0 to r_1, because the top posterior recruits at both strengths, and 0.272757 from r_1 to r_2. At r_0 the frozen profile is a control and not an equilibrium: its purchase and sale earn $\Delta_0(1-M) - k < 0$ and $\rho\Delta_0 m - k < 0$. At r_1, the bound $\rho\kappa\Delta_1 > k$ of Step 2 of the proof of Proposition 2 forces full orders in every candidate, so the frozen tuple is the unique equilibrium there.

When the rival does not see the price, its belief stays at π, and $B_{r_i}(\pi) < c_H$ gives entry ρ after every order at both strengths. At r_0, $\Delta_0 < k$ forces zero orders; at r_1, the bound $\rho\kappa\Delta_1 > k$ holds with $e \equiv \rho$ and forces full orders. The prices are $t_0 + \rho\, w(\mu(x))$, and every condition holds by the computations of Steps 1 and 3 of the proof of Proposition 2 with $e \equiv \rho$. Entry is 0.250000 at both strengths. Table 3 lists all four outcomes.

Preparation cost distributions

For this robustness result, let the preparation cost C be independent of the class, the value, the incumbent's value and the slot, with the distribution

$$\Phi = \rho\Phi^0 + (1-\rho)\Phi^1, \qquad \operatorname{supp}\Phi^0 \subset [0, \bar{c}_0], \qquad \operatorname{supp}\Phi^1 \subset [c_-, c_+].$$

Each component can be atomless or have atoms. A rival prepares when gross profit at its belief exceeds its cost, and it stays out when gross profit falls short. The low component plays the role of the zero cost, and the high component plays the role of c_H.

With a general cost distribution, the preparation rule assigns $\zeta(\bar{P}, C) \in [0,1]$ to each price

and cost, and (E3) reads: $\zeta(\bar{P}, C) = 1$ if $B(\bar{\mu}(\bar{P})) > C$, and $\zeta(\bar{P}, C) = 0$ if $B(\bar{\mu}(\bar{P})) < C$. Entry after the order x is $e(x) = \int \zeta(P(x), C)\, d\Phi(C)$. Every step of Appendix A.2 that does not use the two-point distribution carries over; in particular, Lemma A.3 holds for any entry in $[\rho, 1]$.

Proposition A.3 (Preparation cost distributions)**.** *Let the classes satisfy the class ranking* (1)*, and let F_1 strengthen F_0. Suppose*

$$0 < \bar{c}_0 < B_1(m), \qquad B_0(\pi) < c_- < c_+ < B_1(M), \qquad \Delta_0 < k < \rho\kappa\Delta_1.$$

(i) At F_0, the unique orders are zero, and entry is ρ.

(ii) At F_1, the unique orders are full, and only a purchase recruits the high component. Entry is $\rho + (1-\rho)(b + \pi\lambda)$. High-class ownership rises if and only if $[\rho + (1-\rho)(b+\lambda)]\, W_H(F_1) > \rho\, W_H(F_0)$.

Proof of Proposition A.3. Step 0: consequences of the hypotheses. The class ranking (1) gives $\Delta_i \geq 0$ and $w_L \geq 0$. Since $k > 0$, $k < \rho\kappa\Delta_1$ gives $\Delta_1 > 0$. The support gaps and Proposition 1(a) give $B_1(M) > c_+ > c_- > B_0(\pi) \geq B_1(\pi)$. Since $M > \pi$, $B_1(M) - B_1(\pi) = (M - \pi)(g_{H,1} - g_{L,1}) > 0$, so $g_{H,1} > g_{L,1}$. The proof needs no condition $\Delta_0 > 0$.

Step 1: the floor at the nonzero orders. Fix a candidate in either economy. A price reached by a nonzero order has its posterior in $[m, M]$, so

$$B_{F_i}(\bar{\mu}) \geq B_{F_i}(m) \geq B_1(m) > \bar{c}_0.$$

Every cost in the support of Φ^0 therefore prepares strictly at both nonzero-order prices, giving entry at least ρ after a purchase and after a sale.

Step 2: orders and preparation by the low component. At F_0, entry in $[0, 1]$ gives correct gross returns at most $\Delta_0 < k$, and wrong-signed orders earn at most $-k$. Both classes therefore abstain. Every order and price has posterior π, and $\bar{c}_0 < B_1(m) \leq B_0(\pi) < c_-$ gives entry ρ. At F_1, the nonzero-order floor and the trading bound give correct gross returns at least $\rho\kappa\Delta_1 > k$, so both classes place full orders. Their posteriors are (m, π, M), and each price belief is an average of these. The low component therefore prepares strictly at every price. These order comparisons hold under both tie rules for the informed trader. The price formula and the strict support windows give the equilibrium tuples and outcomes of Proposition A.3.

Step 3: the tuples. At F_0: zero orders, $\mu \equiv \pi$, $\bar{\mu} \equiv \pi$, the single price $t_{0,0} + \rho\, w_0(\pi)$, and $\zeta(\bar{P}, C) = \mathbf{1}\{C \leq B_0(\pi)\}$. At F_1: full orders, posteriors (m, π, M), $\zeta(P(x), C) = \mathbf{1}\{C \leq B_1(\mu(x))\}$, entry $(\rho, \rho, 1)$, prices $t_{0,1} + e(x)\, w_1(\mu(x))$ and $\bar{\mu}(P(x)) = \mu(x)$; truthful bids in both. Equilibrium conditions: (E1) and (E2) hold as in Steps 1 and 3 of the proof of Proposition 2; at F_1, the stated entry is nondecreasing in the posterior and $\Delta_1 > 0$, so the prices differ. (E3): at F_0, $B_0(\pi) < c_-$

excludes the whole high component, and $B_0(\pi) > \bar{c}_0$ includes the whole low component. At F_1, $B_1(M) > c_+$ includes the whole high component, and $B_1(m) < B_1(\pi) \leq B_0(\pi) < c_-$ excludes it at the other two posteriors. Every realized comparison is strict, so both tie rules for the rival agree. (E4): (A.4). (E5): at F_0, the correct gross returns are $\rho\Delta_0(1-\pi)$ and $\rho\Delta_0\pi$, both below k; at F_1, the purchase earns $\Delta_1(1-M) - k > 0$ and the sale $\rho\Delta_1 m - k > 0$. (E6): truthful bids.

Step 4: outcomes. A purchase has probability $b+\lambda$ in class H and b in class L, and only a purchase admits the high component. So $\mathsf{E}_1 = \rho + (1-\rho)(b + \pi\lambda)$, class entry is $\bar{e}_H = \rho + (1-\rho)(b+\lambda)$, and ownership is $\pi\bar{e}_H W_H(F_1)$ against $\pi\rho W_H(F_0)$, as in Step 4 of the proof of Proposition 2. □

The two-point distribution is the case $\Phi^0 = \delta_0$, $\Phi^1 = \delta_{c_H}$.

The third economy and the example band. For F_2, the conditions $\bar{c}_0 < B_2(m)$, $B_2(M) < c_-$ and $k < \rho\kappa\Delta_2$ give full orders by the same bound, include the low component everywhere, and exclude the high component at every posterior, so entry is ρ. Suppose instead that the low component prepares everywhere, that $B_r(\pi) < c_-$, and that the orders are full. The high component then prepares only after a purchase, with probability $\Phi^1(B_r(M))$, so

$$\mathsf{E}(r) = \rho + (1-\rho)(b + \pi\lambda)\,\Phi^1\big(B_r(M)\big).$$

This is the only place where atomlessness enters: for atomless Φ^1, Φ^1 is continuous, and $B_r(M)$ falls continuously in r, so entry is continuous and nonincreasing in strength. In the example, take $\bar{c}_0 = 1/5$ and Φ^1 uniform on $[c_H - \varepsilon, c_H + \varepsilon]$ with $\varepsilon = 1/10$. The margins are $B_1(m) - \bar{c}_0 = 2.166179$, $c_- - B_0(\pi) = 1.095833$ and $B_1(M) - c_+ = 0.117155$. Entry at r_1 is 0.522757 and ownership is 0.324401. At r_2, $c_- - B_2(M) = -0.097311$: the top profit lies inside the band, at distances 0.097311098 and 0.102688902 from its ends, and the formula gives entry 0.382711 and ownership 0.222020 by direct evaluation of the uniform distribution. The return to ρ at r_2 holds for every half-width below $c_H - B_2(M) \approx 0.002689$; at $\varepsilon = 1/1000$ its margin is 0.001689.

Value classes and priors

Scope. For every pair of ranked finite-mean classes with a positive mean gap, a reserve $p > 0$, every interior prior and slot, and $0 < \rho < 1$, the uniform incumbent family $F_r = U[0, r]$ admits a nonempty set, defined by strict inequalities in strengths and costs, on which the hypotheses and ownership conclusion of Proposition 2 hold.

Proposition A.4 (Value classes and priors)**.** *Fix classes that satisfy the class ranking* (1) *with* $\mathbb{E}_H[\theta] > \mathbb{E}_L[\theta]$*, a reserve* $p > 0$*, and any* $\pi, \rho \in (0,1)$ *and slot* (λ, β)*. For uniform incumbents, a*

nonempty open set of strengths and costs (r_0, r_1, r_2, c_H, k) satisfies (H1)–(H3) and the ownership condition of Proposition 2. On this set every conclusion of Proposition 2 holds.

Proof of Proposition A.4. Let $\Gamma(x) = \int_p^x D(y)\,dy$, and write $\bar{\Gamma} = \mathbb{E}_H[\theta] - \mathbb{E}_L[\theta] > 0$ for the total mean gap. Under the class ranking (1), $D \geq 0$, so Γ is continuous and nondecreasing with limit $\bar{\Gamma}$. Let $\theta_* = \inf\{x \geq p : \Gamma(x) > 0\}$; finite means and $\bar{\Gamma} > 0$ give $p \leq \theta_* < \infty$.

Step 1: the uniform incumbent. For $R \sim U[0, r]$, (5) and integration by parts give

$$\Delta_T(r) = \int_p^r \Big(1 - \frac{y}{r}\Big) D(y)\,dy = \frac{1}{r}\int_p^r \Gamma(y)\,dy,$$

since $\Gamma(p) = 0$ and the boundary term at r vanishes. Since Γ is continuous,

$$\Delta_T'(r) = \frac{r\Gamma(r) - \int_p^r \Gamma(y)\,dy}{r^2} \geq \frac{p\,\Gamma(r)}{r^2} > 0 \qquad (r > \theta_*),$$

because $\int_p^r \Gamma \leq (r-p)\Gamma(r)$. Also $\Delta_T(r) \to 0$ as $r \downarrow \theta_*$, since $\Gamma = 0$ on $[p, \theta_*]$. The split identity of Section 3.1 gives $g_H(r) - g_L(r) = \bar{\Gamma} - \Delta_T(r) \geq \bar{\Gamma} - (r-p)\bar{\Gamma}/r = p\bar{\Gamma}/r > 0$, and at $r = \theta_*$, $g_H - g_L = \bar{\Gamma}$. Dominated convergence, with the integrable bound $1 - G_j$, gives continuity of $g_j(r)$ and $W_H(r)$ in r, and $g_j(r) \to 0$ as $r \to \infty$. Since $\theta \geq p$, $W_H(r) = \mathbb{E}_H[\min\{\theta/r, 1\}] \geq p/r > 0$.

Step 2: the construction. Let $e_H = \rho + (1-\rho)(b+\lambda) > \rho$. At θ_*, $B_{\theta_*}(M) - B_{\theta_*}(\pi) = (M - \pi)\bar{\Gamma} > 0$. By continuity choose $r_1 > \theta_*$ close enough to θ_* that $B_{r_1}(M) > B_{\theta_*}(\pi)$ and $e_H W_H(r_1) > \rho W_H(\theta_*)$. Then choose $r_0 \in (\theta_*, r_1)$ close enough to θ_* that $\Delta_T(r_0) < \rho\kappa\Delta_T(r_1)$. Since $B_r(\pi)$ and $W_H(r)$ fall in r, $B_{r_0}(\pi) \leq B_{\theta_*}(\pi) < B_{r_1}(M)$ and $e_H W_H(r_1) > \rho W_H(\theta_*) \geq \rho W_H(r_0)$. Choose c_H in $(B_{r_0}(\pi), B_{r_1}(M))$ and k in $(\Delta_T(r_0), \rho\kappa\Delta_T(r_1))$. Since $g_j(r) \to 0$, choose $r_2 > r_1$ with $B_{r_2}(M) < c_H$. These choices satisfy (H1) to (H3) and the ownership condition.

Step 3: openness and the conclusions. Each condition is a strict inequality between continuous functions of (r_0, r_1, r_2, c_H, k), so a neighborhood of the constructed point satisfies all of them. On this set, the tuples of Proposition 2 satisfy every equilibrium condition and are the unique outcomes. In the point-mass case $D = \mathbf{1}\{\ell \leq y < h\}$, so $\theta_* = \ell$. □

The economies of Table A.4. For a point value θ and $R \sim U[0, r]$, integration over the pieces cut by p, θ and r gives $t(\theta) = \theta - (\theta^2 - p^2)/(2r)$, $g(\theta) = (\theta^2 - p^2)/(2r)$ and $F(\theta) = \theta/r$ for $p \leq \theta \leq r$, and $t(\theta) = r/2 + p^2/(2r)$, $g(\theta) = \theta - t(\theta)$ and $F(\theta) = 1$ for $\theta \geq r$. A uniform class averages these piecewise polynomials over its support, with a cut at r; every piece has degree at most three. Exact integration, with rational bounds on e, gives (H1)–(H2) for the moderate-gap row and (H1)–(H3) for the other four reversal rows, with the margins in Table A.6. Proposition 2 supplies their unique equilibrium outcomes. In the wide classes the class ranking holds with overlapping

Table A.4. Other class distributions and priors.

Panel A. Primitives								
Economy	Low class	High class	π	c_H	k	r_0	r_1	r_2
Moderate gap	1	2	1/2	89/100	1/500	21/20	3/2	
Narrow bands	$U[19/20, 21/20]$	$U[199/20, 201/20]$	1/2	6	1/50	6/5	3	18/5
Low prior	1	10	3/10	9/2	1/100	11/10	3	18/5
High prior	1	10	7/10	36/5	1/100	11/10	3	18/5
Wide classes	$U[3/5, 7/5]$	$U[1, 19]$	7/10	67/10	1/50	1	9/2	11/2
Same means, points	1	10	7/10	67/10	1/50	1	9/2	11/2

Panel B. Outcomes			
Economy	Entry E_0	Entry E_1	High-class ownership $\mathsf{O}_{H,1}$
Moderate gap	0.250000	0.522757	0.324401
Narrow bands	0.250000	0.522757	0.324401
Low prior	0.250000	0.472339	0.194641
High prior	0.250000	0.573175	0.454162
Wide classes	0.250000	0.573175	0.419819
Same means, points	0.250000	0.250000	0.175000

Notes. The slot with $\beta = 0.294655$ and $1 + u = \mathrm{e}$, the reserve $1/2$ and $\rho = 1/4$ in every row. A single number is a point-mass class; $U[\cdot, \cdot]$ is a uniform class. The moderate-gap economy has two strengths. At the third strength every row with three strengths has entry ρ. The moderate-gap row satisfies (H1)–(H2) of Proposition 2; the other four reversal rows also satisfy (H3). The last row fails the recruitment condition at r_1.

supports: $G_L - G_H \geq 0$ everywhere, with equality only outside the common interior of the supports.

The last row keeps the costs, the prior and the strengths of the wide classes and puts point masses at the same means. Its upper margin at r_1 is $B_1(M) - c_H = -0.018122 < 0$. The bounds of Steps 1 and 2 of the proof of Proposition 2 still force zero orders at r_0 and full orders at r_1 and r_2, since their trading margins are positive (Table A.6). Then $B_i(\mu) \leq B_i(M) < c_H$ at every posterior of the higher strengths, and the prior profit at r_0 is below c_H, so entry is ρ at all three strengths. The comparison of the two rows is a convexity statement. For each R, $\theta \mapsto (\theta - \max\{p, R\})_+$ is convex, so a mean-preserving increase in risk within a class weakly raises g_j, and hence gross profit at every belief.

At the example's costs and strengths, the prior $3/10$ gives entry 0.250000 at all three strengths: the signed profit gap at r_1 is $B_{r_1}(M(3/10)) - c_H = -1.390821$. The prior $7/10$ gives entry 1.000000 at r_0, because the prior profit there exceeds the high cost by 0.600833, and entry 0.573175 at r_1. In both cases the bound $\Delta_T(r_0) < k$ forces zero orders at r_0, and the bound on every profile forces full orders at r_1 and r_2.

More than two classes

Let finitely many classes j have priors $\pi_j > 0$ and values above the reserve, and write $t_{j,i}$ and $g_{j,i}$ for the class objects at F_i. Every posterior vector ν that a nonzero order can produce lies in the set

$$\mathcal{M}_\pi = \left\{\nu : \ \frac{\nu_j}{\pi_j} \le (1+u)\frac{\nu_l}{\pi_l} \text{ for all } j, l\right\}.$$

The price reveals only the posterior mean of class proceeds, $\nu \cdot t$. Partition the classes into a low group $\mathcal{L}$ and a high group $\mathcal{H}$, with high-group prior mass $\bar{\pi}$. Write $\bar{g}_{\mathcal{H}}$ and $\bar{g}_{\mathcal{L}}$ for the prior-weighted group means of $g_{j,1}$.

Proposition A.5 (Many classes under separation)**.** *Let the incumbent strengthen from F_0 to F_1. Suppose*

$$\max_j t_{j,0} - \min_j t_{j,0} < k, \qquad \sum_j \pi_j g_{j,0} < c_H,$$

$$\min_{j\in\mathcal{H}} t_{j,1} - \max_{\nu\in\mathcal{M}_\pi} \nu \cdot t_1 > \frac{k}{\rho}, \qquad \min_{\nu\in\mathcal{M}_\pi} \nu \cdot t_1 - \max_{j\in\mathcal{L}} t_{j,1} > \frac{k}{\rho},$$

and $\bar{g}_{\mathcal{H}} > \bar{g}_{\mathcal{L}}$ with $\bar{g}_{\mathcal{L}} + M(\bar{\pi})(\bar{g}_{\mathcal{H}} - \bar{g}_{\mathcal{L}}) > c_H$. Then at F_0 the unique orders are zero, and entry is ρ. At F_1 the unique orders are a purchase in every high-group class and a sale in every low-group class, only a purchase recruits, and entry is $\rho + (1-\rho)(b + \bar{\pi}\lambda) > \rho$. These conclusions hold under either tie rule for the rival.

Proof of Proposition A.5. The tuple lists every σ_j, the posterior vector $\mu(q) = (\mu_j(q))_j$ after every order, the posterior vector at every price, the prices, ζ and the bids. For $q = \pm 1$, $b \le f_j(q) \le b + \lambda$, so every ratio $f_j(q)/f_l(q)$ lies in $[(1+u)^{-1}, 1+u]$. Bayes' rule gives $\mu_j(q)/\pi_j = f_j(q)/\sum_l \pi_l f_l(q)$, whence the posterior vectors at the two nonzero orders belong to $\mathcal{M}_\pi$ for every $0 < \beta < 1/2$.

Payoffs. Competitive prices give $P(q) = t_0 + e(q)\,[\mu(q) \cdot t - t_0]$, and $\mathbb{E}[V_T \mid j, q] = t_0 + e(q)(t_j - t_0)$ as in (A.4). Hence, against the fixed price and entry,

$$U_j(q) = q\, e(q)\,[t_j - \mu(q) \cdot t] - k|q|,$$

with $e(q) \in [\rho, 1]$ because the zero-cost rival prepares after every price. This step needs no price sufficiency.

The weak economy. Every residual satisfies $|t_{j,0} - \mu \cdot t_0| \le \max_l t_{l,0} - \min_l t_{l,0} < k$, so every nonzero order loses, and zero is the unique order of every class. Then $f_j = f_l$ for all j, l, every posterior vector equals the prior vector π, and gross profit is $\sum_j \pi_j g_{j,0} < c_H$, so entry is ρ. The tuple has zero orders, the prior posterior at every order and price, the single price $t_{0,0} + \rho\,(\pi \cdot t_0 - t_{0,0})$, $\zeta \equiv 0$, and truthful bids. Each condition holds by these computations.

The strong economy, orders. Under every candidate, a high-group class j earns from a purchase at least $\rho\,[\min_{l\in\mathcal{H}} t_{l,1} - \max_{\nu\in\mathcal{M}_\pi} \nu\cdot t_1] - k > 0$, and from a sale at most $-k$ minus a nonnegative term, since $t_{j,1} - \mu\cdot t_1 > 0$. A low-group class earns from a sale at least $\rho\,[\min_{\nu\in\mathcal{M}_\pi} \nu\cdot t_1 - \max_{l\in\mathcal{L}} t_{l,1}] - k > 0$, and from a purchase at most $-k$. Zero pays zero. So every equilibrium has purchases in $\mathcal{H}$ and sales in $\mathcal{L}$.

The strong economy, the tuple. With these orders, the group posterior $\mu_\mathcal{H}$ of $\mathcal{H}$ follows the two-class formulas with prior $\bar\pi$: $\mu_\mathcal{H} = (m(\bar\pi), \bar\pi, M(\bar\pi))$ after $(-1, 0, 1)$. Within each group every class has the same order distribution, so the class proportions stay at the prior proportions: $\mu_j(\mu_\mathcal{H}) = \mu_\mathcal{H}\pi_j/\bar\pi$ for $j \in \mathcal{H}$ and $\mu_j(\mu_\mathcal{H}) = (1-\mu_\mathcal{H})\pi_j/(1-\bar\pi)$ for $j \in \mathcal{L}$. Gross profit at $\mu_\mathcal{H}$ is $\bar g_\mathcal{L} + \mu_\mathcal{H}(\bar g_\mathcal{H} - \bar g_\mathcal{L})$, increasing in $\mu_\mathcal{H}$. At $\mu_\mathcal{H} = \bar\pi$ it is $\sum_j \pi_j g_{j,1} \le \sum_j \pi_j g_{j,0} < c_H$ by Proposition 1(a); at $\mu_\mathcal{H} = M(\bar\pi)$ it exceeds c_H by hypothesis. So only a purchase recruits, and entry is $(\rho, \rho, 1)$. The prices are $P = t_{0,1} + e(\mu_\mathcal{H})[\bar t_\mathcal{L} - t_{0,1} + (\bar t_\mathcal{H} - \bar t_\mathcal{L})\mu_\mathcal{H}]$ with group means $\bar t$. Separation gives $\bar t_\mathcal{H} > \max_\nu \nu\cdot t_1 \ge \min_\nu \nu\cdot t_1 > \bar t_\mathcal{L}$, so with e nondecreasing in $\mu_\mathcal{H}$ and $e \ge \rho$, the three prices differ, each price reveals $\mu_\mathcal{H}$ and the whole vector, and $\bar\mu(P) = \mu(\mu_\mathcal{H})$. Each condition then holds by direct computation: (E1) and (E2) by the formulas above, (E3) by the strict profit comparisons, (E4) by the price formula, (E5) by the strict payoff bounds, (E6) by truthful bids. A purchase has probability $b + \bar\pi\lambda$, so entry is $\rho + (1-\rho)(b + \bar\pi\lambda) > \rho$. Every comparison is strict, so both tie rules for the rival give this outcome. □

The four-class economy. The economy has the example slot, costs, reserve and first two strengths, and four point-mass classes with prior $1/4$ each. The low group has the values 1 and $1001/1000$, and the high group has the values $9999/1000$ and 10. For posterior vectors of the form $\nu_j = \pi_j\iota_j/\sum_l \pi_l\iota_l$ with $1 \le \iota_j \le 1 + u$, the mean $\nu\cdot t$ is linear-fractional in each weight, so its extremes over $\mathcal{M}_\pi$ occur at the vertices of the weight box. Evaluation at every vertex gives the separation distance $\varsigma = 0.178871$, the trading margin $\rho\varsigma - k = 0.024718$, the weak range margin 0.003333, the weak cost margin 1.195875, the group profit gap 8.332667 and the upper profit margin 0.216834.

A price collision at three classes. Without separation, the price can pool orders with different posterior vectors. Take the slot of the example, $R \sim U[0,8]$, $p = 1$, $\rho = 1/4$, $c_H = 10$, $k = 1/8$, priors $(1/4, 1/2, 1/4)$, and the classes $G_1 = \delta_p$, $G_2 = (1-\psi)\delta_p + \psi\delta_{10}$, $G_3 = \delta_{10}$ with $\psi = (1 + 2b_0 + \lambda)/2 = 0.940709$. Since $4\beta > 1$, $1 - \psi = (1-\lambda)(4\beta - 1)/2 > 0$, so the classes are strictly ordered. The auction gives $t_0 = p(1 - p/r)$,

$$t_1 = p, \quad t_3 = \frac{r}{2} + \frac{p^2}{2r}, \quad t_2 = p + \psi(t_3 - p), \qquad g_1 = 0, \quad g_3 = h - t_3, \quad g_2 = \psi g_3,$$

where $h = 10$ and $r = 8$. The tuple is $\sigma_j = \delta_{j-2}$ (class 1 sells, class 2 abstains, class 3 buys), Bayes posteriors at each order, entry ρ at every price, prices $P(q) = t_0 + \rho(\mu(q) \cdot t - t_0)$, the pooled belief at the common price of the orders 0 and 1, and truthful bids. Every class profit is at most $g_3 < c_H$, and class 3 has positive probability at every price, so the high cost never prepares and the zero cost prepares strictly. Normalize proceeds to $(0, \psi, 1)$. The conditional means at the orders 0 and 1 are

$$\bar{t}(0) = \frac{b_0(\pi_2\psi + \pi_3) + \lambda\pi_2\psi}{b_0 + \lambda\pi_2}, \qquad \bar{t}(1) = \frac{b(\pi_2\psi + \pi_3) + \lambda\pi_3}{b + \lambda\pi_3},$$

and $\bar{t}(0) = \bar{t}(1)$ exactly when $\psi = (2b + 3b_0 + 2\lambda)/[2(2b + b_0 + \lambda)]$, which equals the stated ψ because $2b + b_0 + \lambda = 1$. The posterior vectors differ: $(0.154664, 0.690671, 0.154664)$ after the order 0 and $(0.174878, 0.349755, 0.475367)$ after the order 1, with the common price 1.522107. The sale has the distinct price 1.292045. Against the fixed tuple, $U_j(q) = q\rho[t_j - \mu(q) \cdot t] - k|q|$, and each class's chosen order beats its best alternative by 0.260795, 0.020627 and 0.024768. The collision is a knife-edge in ψ: a change of $1/1000$ in either direction separates the two prices and keeps every choice strict.

Complementary private signals

Proposition A.6 (Complementary private signals)**.** *Fix ranked finite-mean classes, a reserve $p > 0$, continuous distributions $F_1 \leq F_0$, prior one half, the two-point preparation cost with $0 < \rho < 1$, and a slot $0 < \lambda < 1$, $0 < \beta < 1/2$. Conditional on class j, the signals S_I, S_C, the rival's value and the incumbent's value are mutually independent, with value distributions G_j, F_i. The cost and slot draws are independent of these variables. The binary signals have accuracies $1/2 < \alpha_I, \alpha_C < 1$:*

$$\Pr(S_I = + \mid H) = \Pr(S_I = - \mid L) = \alpha_I, \qquad \Pr(S_C = + \mid H) = \Pr(S_C = - \mid L) = \alpha_C.$$

The informed trader observes S_I; the rival observes S_C and the price. Define

$$M^I = (1 - \alpha_I) + (2\alpha_I - 1)M(1/2), \quad m^I = (1 - \alpha_I) + (2\alpha_I - 1)m(1/2),$$
$$\mu_+^C(\mu) = \frac{\alpha_C\mu}{\alpha_C\mu + (1 - \alpha_C)(1 - \mu)}, \quad \mu_-^C(\mu) = \frac{(1 - \alpha_C)\mu}{(1 - \alpha_C)\mu + \alpha_C(1 - \mu)},$$
$$\bar{\Delta}_P(F) = \Delta_T(F) + (1 - \rho)(2\alpha_C - 1)[t_H(F) - t_0(F)].$$

Suppose that

$$B_{F_0}(\alpha_C) < c_H < B_{F_1}(\mu_+^C(M^I)), \qquad (2\alpha_I - 1)\bar{\Delta}_P(F_0) < k < (2\alpha_I - 1)\rho\, m(1/2)\Delta_T(F_1).$$

Under all four tie rules, the unique orders at F_0 are abstention after either signal of the informed trader, with entry ρ and high-class ownership $\rho W_H(F_0)/2$. At F_1, the unique orders are a purchase after + and a sale after −; the public posteriors after $(-1, 0, 1)$ are $(m^I, 1/2, M^I)$, the three prices differ, and entry strictly exceeds ρ.

At F_1, write $\zeta_\pm(\mu)$ for high-cost preparation following $S_C = \pm$ at public belief μ. For $\mu_\pm^C(\mu)$ strictly above or below the preparation threshold it is respectively one or zero; at equality the chosen tie rule for the rival applies. Conditional entry is

$$e_H(\mu) = \rho + (1-\rho)[\alpha_C\zeta_+(\mu) + (1-\alpha_C)\zeta_-(\mu)],$$
$$e_L(\mu) = \rho + (1-\rho)[(1-\alpha_C)\zeta_+(\mu) + \alpha_C\zeta_-(\mu)].$$

Its class average $\bar{e}_H = \sum_x f_H(x)e_H(\mu(x))$ gives ownership $\mathsf{O}_{H,1} = \bar{e}_H W_H(F_1)/2$, where

$$f_H = (b + (1-\alpha_I)\lambda, b_0, b + \alpha_I\lambda), \qquad f_L = (b + \alpha_I\lambda, b_0, b + (1-\alpha_I)\lambda).$$

If also $B_{F_1}(\mu_-^C(M^I)) < c_H$, exactly the pair $(X, S_C) = (1, +)$ recruits, and

$$\bar{e}_H = \rho + (1-\rho)\alpha_C(b + \alpha_I\lambda), \qquad \bar{e}_L = \rho + (1-\rho)(1-\alpha_C)[b + (1-\alpha_I)\lambda].$$

In general high-class ownership rises exactly when $\bar{e}_H W_H(F_1) > \rho W_H(F_0)$; in particular it rises if $W_H(F_1) = 1$, or if decisive advantage holds at F_1.

Proof. Payoff bounds. The cost window and stochastic dominance imply $B_{F_1}(\mu_+^C(M^I)) > B_{F_1}(\alpha_C)$, hence $g_H(F_1) > g_L(F_1)$. The profit gap at F_0 is at least as large by Lemma A.5, and the trading window gives $\Delta_T(F_1) > 0$. For a candidate, let $f_\pm(x) = (1-\lambda)\eta(x) + \lambda\sigma_\pm(x)$ and $\nu_I(x) = f_+(x)/[f_+(x) + f_-(x)]$. The two signals of the informed trader have equal prior probabilities, and conditional independence gives

$$\mu(x) = (1-\alpha_I) + (2\alpha_I - 1)\nu_I(x).$$

At either nonzero order Lemma A.1(b), applied to the informed trader's signal, gives $m(1/2) \le \nu_I(x) \le M(1/2)$. At a price the rival combines its signal with the public belief using $\mu_\pm^C$ from the statement. Write its preparation probabilities there as $\zeta_\pm$. The signal likelihood ratios are strictly ordered, so $\zeta_+ \ge \zeta_-$, under either tie rule for the rival. Consequently $e_H \ge e_L \ge \rho$, and

$$e_H - e_L = (1-\rho)(2\alpha_C - 1)(\zeta_+ - \zeta_-).$$

With $w_j = t_j - t_0$, conditional independence gives terminal payoff $t_0 + e_j w_j$ in class j. Competitive

pricing is therefore $P(x) = t_0 + e_L w_L + \mu(x)\Delta_P$, where

$$\Delta_P = e_H w_H - e_L w_L = e_L \Delta_T + (e_H - e_L) w_H, \qquad \rho\Delta_T \le \Delta_P \le \bar{\Delta}_P. \tag{A.17}$$

These bounds apply at every candidate price. If two orders share a price, the two signal-contingent preparation probabilities agree, so their class entries agree. At F_1, the positive lower bound in (A.17) then gives equal public posteriors. Bayes' rule at that price gives the common posterior.

Subtracting the price from the terminal payoff after each signal of the informed trader gives

$$\begin{aligned} U_+(1) &= (2\alpha_I - 1)[1 - \nu_I(1)]\Delta_P(1) - k, \\ U_-(-1) &= (2\alpha_I - 1)\nu_I(-1)\Delta_P(-1) - k. \end{aligned}$$

The wrong-signed orders earn respectively $-(2\alpha_I - 1)[1 - \nu_I(-1)]\Delta_P(-1) - k$ and $-(2\alpha_I - 1)\nu_I(1)\Delta_P(1) - k$, at most $-k$. Zero pays zero. Every correct gross return is at most $(2\alpha_I - 1)\bar{\Delta}_P$ and at least $(2\alpha_I - 1)m(1/2)\rho\Delta_T$.

The weaker distribution. The upper trading bound at F_0 is below k, so abstention after either signal is forced under both tie rules for the informed trader. All order and price posteriors are one half. The two private posteriors are α_C and $1 - \alpha_C$; both profits are below c_H. Thus entry is ρ, all orders share price $t_0 + \rho(w_H + w_L)/2$, and truthful bids complete the tuple. The displayed payoff bounds give strict optimality of abstention. Preparation is independent of both values given class, so high-class ownership is $\rho W_H(F_0)/2$.

The stronger distribution. The lower trading bound at F_1 exceeds k, so the unique orders are a purchase after $S_I = +$ and a sale after $S_I = -$. Then $f_+ = (b, b_0, b + \lambda)$, $f_- = (b + \lambda, b_0, b)$, $\nu_I = (m(1/2), 1/2, M(1/2))$, and the public posteriors are $(m^I, 1/2, M^I)$. Choose $\zeta_\pm$ by the prescribed preparation rule at these beliefs. Each is nondecreasing across the three distinct beliefs, so both e_H and e_L are nondecreasing. If $\mu_2 > \mu_1$,

$$\begin{aligned} P(\mu_2) - P(\mu_1) = {} & (\mu_2 - \mu_1)\Delta_P(\mu_1) \\ & + \mu_2[e_H(\mu_2) - e_H(\mu_1)]w_H \\ & + (1 - \mu_2)[e_L(\mu_2) - e_L(\mu_1)]w_L > 0. \end{aligned}$$

The first term is positive and the others nonnegative. The prices are therefore distinct, each reveals its public posterior, and combining it with the private signal gives the stipulated preparation. The price formula, strict optimal-order bounds and truthful bids establish the other equilibrium conditions. At the purchase price $B_{F_1}(\mu_+^C(M^I)) > c_H$; the event $(X, S_C) = (1, +)$ has positive probability in each class. Thus each class average of entry, and total entry, strictly exceeds ρ.

Ownership and the recruiting pair. Conditional on class, the signals, slot and cost select

preparation independently of (θ, R). Therefore $\mathsf{O}_{H,1} = \bar{e}_H W_H(F_1)/2$, with the class-specific order distributions in the statement. If $B_{F_1}(\mu_-^C(M^I)) < c_H$, the rival's unfavorable signal never recruits. After a sale or abstention, the public belief is at most one half, and $B_{F_1}(\mu_+^C(1/2)) = B_{F_1}(\alpha_C) \leq B_{F_0}(\alpha_C) < c_H$. Thus exactly $(X, S_C) = (1, +)$ recruits. Its probabilities are $\alpha_C(b + \alpha_I\lambda)$ in class H and $(1-\alpha_C)[b + (1-\alpha_I)\lambda]$ in class L, giving the stated averages. Comparing ownership at the two distributions gives the exact condition. If $W_H(F_1) = 1$, dominance gives $W_H(F_0) = 1$ and $\bar{e}_H > \rho$ gives the strict rise. Recruitment at F_1 gives $c_H < B_{F_1}(\mu_+^C(M^I)) < g_H(F_1)$, because $\mu_+^C(M^I) < 1$ and $g_H(F_1) > g_L(F_1)$; decisive advantage at F_1 then gives $W_H(F_1) = 1$, as in Lemma A.6. □

The signal example. Take point classes $h = 10$, $\ell = 1$, reserve $p = 1/2$, uniform incumbents on $[0, r]$, $(r_0, r_1) = (1.1, 2.3)$, $(\rho, c_H, k) = (0.85, 7.14, 0.015)$, and $(\alpha_I, \alpha_C) = (0.70, 0.75)$, at the slot $\beta = 0.294655$, $1 + u = \mathrm{e}$. Let $m = m(1/2)$ and $M = M(1/2)$. The four conditions of the proposition and the one-pair condition have the margins 0.052500, 0.045265, 0.001455, 0.018594 and 4.159475. The formulas above give entry 0.850000 and 0.879797, and ownership 0.425000 and 0.449238. With an uninformative signal for the rival ($\alpha_C = 1/2$), the top public profit stays below c_H at both strengths, with margins 1.469972860 and 1.862796856, so entry is ρ; the bounds 0.013181818 and 0.018594291 force zero and full orders, and entry is 0.850000 at both strengths. When the informed trader observes the class and the rival has no signal, the margins 0.010454545, 0.068985729, 2.301363636 and 0.666013879 give zero orders, full orders and entry 0.850000 at both strengths. On the grid of accuracies, six pairs satisfy the conditions, all with $\alpha_C > \alpha_I$; the smallest margin among them is 0.000100000.

A slot with a more accurate signal for the rival. Keep the signal example's primitives and set $(\lambda, \beta) = (1/3, 3/10)$; then $M = 8/11$ and $m = 3/11$. The five margins are

$$\begin{aligned} c_H - B_{r_0}(\alpha_C) &= 0.052500 > 0, \\ B_{r_1}(\mu_+^C(M^I)) - c_H &= 0.037038 > 0, \\ k - (2\alpha_I - 1)\bar{\Delta}_P(r_0) &= 0.001455 > 0, \\ (2\alpha_I - 1)\rho m \Delta_T(r_1) - k &= 0.019067 > 0, \\ c_H - B_{r_1}(\mu_-^C(M^I)) &= 4.171359 > 0. \end{aligned}$$

Thus the hypotheses allow $\alpha_C > \alpha_I$.

The example. In the example the conditions cannot hold with $\alpha_C \geq \alpha_I$. The strong bound needs $2\alpha_I - 1 > k/[m\rho\Delta_T(r_1)] = 0.446193819$. If $\alpha_C \geq \alpha_I$, then $2\alpha_C - 1 \geq 2\alpha_I - 1$, and the weak gross

bound is

$$(2\alpha_I - 1)\bar{\Delta}_P(r_0) \geq \frac{k}{m\rho\Delta_T(r_1)}\left[\Delta_T(r_0) + (1-\rho)\frac{kw_H(r_0)}{m\rho\Delta_T(r_1)}\right].$$

The right side exceeds k by 0.049030. With an accurate signal for the informed trader and a weak signal for the rival, $(\alpha_I, \alpha_C) = (0.94, 0.505)$, the five margins are 1.150917, 0.020731, 0.002611, 0.019445 and 0.048822. Entry rises from 0.250000 to 0.386933. Without the rival's signal, the top public profit at r_1 stays below c_H, with margin 0.013904, the bounds 0.005333333 and 0.019444742 fix the orders, and entry is 0.250000 at both strengths. Both sources are therefore needed for the rise. Every comparison is strict, so all four pairs of tie rules give the same outcomes. At a higher accuracy of the informed trader's signal, the public posterior alone can recruit, and a pair can satisfy the proposition without this complementarity; the margin $c_H - B_{r_1}(M^I)$ is the condition that separates the two.

The example with the signal example's accuracies. In the example with $(\alpha_I, \alpha_C) = (0.70, 0.75)$, three of the four conditions fail, and each equilibrium has a mixed order: both signal types mix at r_0, and the type with $S_I = -$ mixes at r_1. After wrong-signed orders are excluded, write $s_+ = \lambda\sigma_+(1)$ and $s_- = \lambda\sigma_-(-1)$; the purchase return depends only on s_+, and the sale return only on s_-, with $1 - \nu_I(1) = b/(2b + s_+)$ and $\nu_I(-1) = b/(2b + s_-)$.

- *At r_0.* Under every profile $B_{r_0}(\mu_+^C(m^I)) > c_H$ and $B_{r_0}(\mu_-^C(M^I)) < c_H$, with margins 0.363883929 and 2.755550596. So $\zeta_+ = 1$ and $\zeta_- = 0$ at every price, $e_H = 13/16$, $e_L = 7/16$, and $\Delta_P = 311/1920$ at every price. Each correct gross return is $(2\alpha_I - 1) \cdot \frac{b}{2b+s_\pm} \cdot 311/1920$, strictly decreasing in its rate, above k at rate zero (margin 0.012395833) and below k at rate λ (margin 0.002574837). The unique zero-profit rates are $s_+ = s_- = 119/96 \cdot b < \lambda$ (margin $u - 119/96 = 0.478698495$). Entry is 0.625000 and ownership 0.406250.

- *At r_1.* A rival with $S_C = -$ never recruits (margin 3.155195358). A purchase has public posterior at least $1/2$, and $B_{r_1}(\alpha_C) > c_H$ (margin 0.375000000), so $S_C = +$ recruits after a purchase, $\Delta_P = 137/192$ there, and the purchase earns at least $(2\alpha_I - 1)\, m\, \Delta_P - k = 0.056760364 > 0$: $s_+ = \lambda$. After a sale, $S_C = +$ recruits exactly when $\mu(-1) \geq 47/106$, that is $s_- \leq 15/19 \cdot b$. On that range the public belief after a sale is at least $47/106$, and $\Delta_P \geq \rho\Delta_T(r_1)$, so the sale earns at least the return at that belief with entry ρ, $19/795$, which exceeds k by 0.003899371; so no equilibrium has $s_- \leq 15/19 \cdot b$, and no tie mixture exists at the public belief $47/106$. Above that rate, $e_H = e_L = \rho$ at the sale price. The net sale payoff $U_-(-1) = (2\alpha_I - 1)\rho\Delta_T(r_1)b/(2b + s_-) - k$ strictly decreases with s_-. Its unique zero-profit rate is $s_- = 4/3 \cdot b < \lambda$ (margin 0.384948495), with $\mu(-1) = 21/50$, below $47/106$ by 0.023396226. After the zero order $S_C = +$ recruits (margin 0.219556089). Only a sale blocks recruitment after $S_C = +$; a sale has probability $b + (1 - \alpha_I)s_-$ in class H and $b + \alpha_I s_-$ in class

L. Thus $\bar{e}_H = \rho + (1-\rho)\alpha_C[1 - b - (1-\alpha_I)s_-]$ and $\bar{e}_L = \rho + (1-\rho)(1-\alpha_C)[1 - b - \alpha_I s_-]$. Hence entry is $(\bar{e}_H + \bar{e}_L)/2 = 0.512521$ and ownership $\bar{e}_H/2 = 0.329227$.

At r_0, the informed trader mixes between trade and abstention after either signal, and it trades with probability 0.721409. At r_1, it buys with probability one after a favorable signal, and after an unfavorable signal it sells with probability 0.775969. Each mixed tuple is unique when an indifferent informed trader may mix, with either tie rule for the rival, by the strict monotonicity of each return and the endpoint signs. When an indifferent informed trader abstains, neither strength has an equilibrium, because the unique qualifying rates are interior. Without the rival's signal, entry is ρ under every profile at both strengths, zero orders are forced at r_0, and at r_1 the unique rates are $s_+ = s_- = 4/3 \cdot b$; this control also requires that an indifferent informed trader may mix, since the gross return at zero, 0.033333333, exceeds k and the return at full orders, 0.017929428, falls short of it.

A.7 Welfare and sale terms

This subsection states and proves Propositions A.7, A.8 and A.9, and it proves the statements of Section 7 that rest on them.

Access to prices

Hold the incumbent at F_1. Compare the equilibrium of Proposition 2 with the equilibrium of an economy in which the rival does not see the price. All other primitives stay fixed, and market makers set competitive prices in both economies. Write $\mathcal{R}$ for expected target proceeds and $\mathcal{W}$ for expected acquisition surplus net of preparation costs. The superscript 1 marks price access, and 0 marks the hidden price. Acquisition surplus excludes transfers between bidders, shareholders and traders.

Proposition A.7 (Access to prices)**.** *Let the hypotheses of Proposition 2 hold. At F_1, both economies have unique full orders. Price access raises expected target proceeds and acquisition surplus net of preparation costs:*

$$\mathcal{R}^1 - \mathcal{R}^0 = (1-\rho)\big\{\pi(b+\lambda)(t_H - t_0) + (1-\pi)\, b\, (t_L - t_0)\big\} > 0,$$
$$\mathcal{W}^1 - \mathcal{W}^0 = (1-\rho)(b + \pi\lambda)\big[B_1(M) - c_H + pF_1(p)\big] > 0.$$

Proof of Proposition A.7. The two tuples. The access tuple is $\mathcal{E}_{\mathrm{F}}(F_1)$ of the proof of Proposition 2. The hidden tuple has full orders, the posteriors (m, π, M) at the orders, the rival's belief π at every price because the rival sees neither the price nor the order, $\zeta \equiv 0$, entry ρ after every order, the prices $P^0(x) = t_0 + \rho\, w(\mu(x))$, and truthful bids. Here and below every object is at F_1.

The conditions of the hidden tuple. (E1): the full-order likelihoods. (E2): the rival's belief is its prior. (E3): $B_1(\pi) \le B_0(\pi) < c_H$, strictly. (E4): (A.4) with $e \equiv \rho$. (E5): with $e \equiv \rho$, the purchase earns $\rho\Delta_1(1-M) - k \ge \rho\kappa\Delta_1 - k > 0$ and the sale $\rho\Delta_1 m - k > 0$, and the wrong-signed orders earn at most $-k$. (E6): truthful bids. These bounds hold for every candidate of the hidden economy, so its orders are unique, and the tuple is unique. The access economy is unique by Proposition 2.

The proceeds gain. Couple the two economies on the class, the value, the cost, the slot identity and the liquidity order. The orders coincide at every realization, so the trading costs coincide. Access adds high-cost preparation exactly after the order 1, which has probability $b + \lambda$ in class H and b in class L. With entry, expected proceeds rise from t_0 to t_j. Hence

$$\mathcal{R}^1 - \mathcal{R}^0 = (1-\rho)\{\pi(b+\lambda)w_H + (1-\pi)\, b\, w_L\}.$$

Each term is nonnegative, and $w_H = w_L + \Delta_1 \ge \Delta_1 > 0$, so the gain is positive.

The surplus gain. Let $\mathcal{V}_0(R) = R\mathbf{1}\{R \ge p\}$ be the allocation value without the rival. With truthful bids, the target goes to the highest admissible value, so with an admissible rival of value θ the allocation value is $\max\{\mathcal{V}_0(R), \theta\}$, and

$$\max\{\mathcal{V}_0(R), \theta\} - \mathcal{V}_0(R) = (\theta - \max\{p, R\})_+ + p\mathbf{1}\{R < p\}. \tag{A.18}$$

For $R < p$, both sides equal θ, since $\theta \ge p$; for $R \ge p$, both sides equal $(\theta - R)_+$. Average over R and over the class at the posterior M of the order 1: one added preparation adds $B_1(M) + pF_1(p)$ to allocation value and costs c_H. Added preparations have probability $(1-\rho)\Pr(X = 1) = (1-\rho)(b + \pi\lambda)$. Hence $\mathcal{W}^1 - \mathcal{W}^0 = (1-\rho)(b+\pi\lambda)[B_1(M) - c_H + pF_1(p)] > 0$, because $B_1(M) > c_H$. Transfers between bidders, shareholders and traders cancel from this sum. □

The example, the transfer and the trade scale. At r_1, proceeds are 0.872531 with access and 0.614583 without, a gain of 0.257948; net surplus is 2.634044 and 2.552083, a gain of 0.081960. If the execution cost is a resource cost, both economies lose λk, and the difference is unchanged. In the access economy the informed trader arrives with probability λ; class H buys with entry one after a purchase, and class L sells with entry ρ after a sale. By the payoffs of Lemma A.3 before the cost k, its expected gross profit is

$$G = \lambda\Delta_1\{\pi(1-M) + (1-\pi)\rho m\} = 0.037665.$$

Competitive prices give $\mathbb{E}[V_T - P(X) \mid X] = 0$, so $\mathbb{E}[X(V_T - P(X))] = 0$. The informed and the liquidity orders partition this expectation, so liquidity traders lose G in expected gross terms. Measure one order unit in shares per share of float, and keep the same signal experiment. The

economy without an informed trader has the posterior π after every order and entry ρ, so its proceeds equal $\mathcal{R}^0$. If initial shareholders bear the whole transfer, their total gain over that economy is $\mathcal{R}^1 - \mathcal{R}^0$ minus the order unit times G. It is positive exactly when the order unit is below $(\mathcal{R}^1 - \mathcal{R}^0)/G = 6.848394$. A deterministic external dividend raises V_T and every price by its amount. It leaves $V_T - P$, every order payoff, the rival's information and entry unchanged, and it enters neither proceeds of the sale nor allocation surplus. With research at the cap, Appendix A.5 gives the same gains.

Access outside the recruitment region. Suppose instead $B_1(m) < c_H < B_1(\pi)$ and $k < \rho\kappa\Delta_1$. Without the price, the rival's belief is π, so every rival prepares, and entry is one after every order. With the price, the posteriors are (m, π, M), and entry is $(\rho, 1, 1)$. In both economies the bound on every profile forces full orders, so the orders, the posteriors and the trading costs coincide, and each tuple satisfies (E1) to (E6) as above. Access removes high-cost preparation exactly after a sale, which has probability b in class H and $b + \lambda$ in class L. Proceeds change by $-(1-\rho)\{\pi b\, w_H + (1-\pi)(b+\lambda)\, w_L\} < 0$, and surplus by $-(1-\rho)[b+(1-\pi)\lambda]\,[B_1(m) - c_H + pF_1(p)]$. Keep the strong incumbent of the example and set $c_H = 12/5$. Then $B_1(m) - c_H = -0.033821$ and $B_1(\pi) - c_H = 1.891667$, and access changes proceeds by -0.173917 and surplus by -0.013505.

The payment rule

Consider a verifiable-value institution with zero reserve. After diligence, values are verifiable, and a sale to the next bidder at its value is an enforceable fallback. The seller and the winner divide the surplus above the fallback by Nash bargaining, with seller weight $\omega \in [0, 1)$. Without a rival, the seller receives ωR.

Proposition A.8 (The payment rule)**.** *Let the rival's values be the point masses ℓ and h, and let the incumbent distribution have support $[0, \bar{r}]$ with $0 < \ell < \bar{r} < h$. The transfer, the rival's expected profit and the spread are*

$$\begin{aligned} T_\omega(R, \theta) &= (1 - \omega) \min\{R, \theta\} + \omega \max\{R, \theta\}, \\ g_{\theta,\omega} &= (1 - \omega)\, \mathbb{E}[(\theta - R)_+], \\ \Delta_\omega &= \omega(h - \ell) + (1 - 2\omega)\, \mathbb{E}[(R - \ell)_+]. \end{aligned}$$

A stronger incumbent weakly lowers the rival's profit. Its effect on the spread weakly has the sign of $1 - 2\omega$. The effect is strict when $\mathbb{E}[(R - \ell)_+]$ changes strictly and $\omega \neq 1/2$. At $\omega = 1/2$, the spread equals $(h - \ell)/2$ at every incumbent distribution.

Proof of Proposition A.8. The continuation. After preparation, the bidder with the largest value, $\max\{R, \theta\}$, acquires the target, and the next bidder's value, $\min\{R, \theta\}$, is the enforceable fallback;

without a rival, the fallback is zero. For $\max\{R,\theta\} > \min\{R,\theta\}$ and $0 < \omega < 1$, Nash bargaining maximizes $(T_\omega - \min\{R,\theta\})^\omega(\max\{R,\theta\} - T_\omega)^{1-\omega}$ over $\min\{R,\theta\} \le T_\omega \le \max\{R,\theta\}$. The logarithm has derivative $\omega/(T_\omega - \min\{R,\theta\}) - (1-\omega)/(\max\{R,\theta\} - T_\omega)$ and a strictly negative second derivative, so the unique optimum solves $\omega(\max\{R,\theta\} - T_\omega) = (1-\omega)(T_\omega - \min\{R,\theta\})$, that is $T_\omega = (1-\omega)\min\{R,\theta\} + \omega\max\{R,\theta\}$. At $R = \theta$, feasibility fixes $T_\omega = \theta$, and at $\omega = 0$ the next bidder accepts the fallback, so $T_\omega = \min\{R,\theta\}$. Hence the transfer is $T_\omega(R,\theta) = (1-\omega)\min\{R,\theta\} + \omega\max\{R,\theta\}$.

Profit and spread. A rival wins when $\theta > R$ and keeps $\theta - T_\omega = (1-\omega)(\theta - R)$, so $g_{\theta,\omega} = (1-\omega)\mathbb{E}[(\theta - R)_+]$. For $R \le \ell$, both classes win, and $T_\omega(R,h) - T_\omega(R,\ell) = \omega(h-\ell)$. For $\ell < R < h$, the high class wins and the low class loses, and $T_\omega(R,h) - T_\omega(R,\ell) = (1-\omega)R + \omega h - (1-\omega)\ell - \omega R = \omega(h-\ell) + (1-2\omega)(R-\ell)$. The support lies below h, so the average over the two cases gives $\Delta_\omega = \omega(h-\ell) + (1-2\omega)\mathbb{E}[(R-\ell)_+]$.

Signs. The function $(\theta - R)_+$ is nonincreasing in R, and $(R-\ell)_+$ is nondecreasing. A first-order stochastic strengthening of the incumbent therefore weakly lowers $g_{\theta,\omega}$ and weakly moves $\mathbb{E}[(R-\ell)_+]$ up, so the spread moves with the sign of $1-2\omega$, strictly when $\mathbb{E}[(R-\ell)_+]$ rises strictly and $\omega \ne 1/2$. At $\omega = 1/2$, the coefficient $1-2\omega$ is zero, and $\Delta_\omega = (h-\ell)/2$. □

For $R \sim U[0,r]$, $\mathbb{E}[(R-\ell)_+] = (r-\ell)^2/(2r) = \Delta_T(r)$, so $\Delta_\omega(r) = \omega(h-\ell) + (1-2\omega)\Delta_T(r)$. In the example, $\Delta_\omega(r_1) - \Delta_\omega(r_0) = (1-2\omega)[\Delta_T(r_1) - \Delta_T(r_0)] = 13/20\,(1-2\omega)$, which is zero exactly at $\omega = 1/2$. The proposition fixes the acquisition stage after every history.

Reserves

At a general reserve, a class value below p gives no admissible bid. For $R \sim U[0,r]$ and a point value θ, integration over $R < p$, $p \le R < \theta$ and $R \ge \theta$ gives the kernel

$$\big(t(\theta),\, g(\theta),\, F(\theta)\mathbf{1}\{\theta \ge p\}\big) = \begin{cases} (t_0, 0, 0), & \theta < p, \\ (p,\, \theta - p,\, 1), & r \le p \le \theta, \\ \big(\theta - (\theta^2 - p^2)/(2r),\, (\theta^2 - p^2)/(2r),\, \theta/r\big), & p \le \theta \le r, \\ \big(r/2 + p^2/(2r),\, \theta - r/2 - p^2/(2r),\, 1\big), & p \le r \le \theta, \end{cases} \tag{A.19}$$

with $t_0 = p(1 - p/r)_+$. The pieces agree at $\theta = r$ and at $p = r$. A class distribution averages the kernel. Both $t(\theta)$ and $g(\theta)$ are nondecreasing in θ, and $t(\theta) \ge t_0$; when $G_H(y) \le G_L(y)$ for every y, $\Delta_T \ge 0$, $w_L \ge 0$ and $g_H \ge g_L \ge 0$. For $\ell < p < r < h$ with point values, $t_L = t_0 = p - p^2/r$, $g_L = 0$, $t_H = r/2 + p^2/(2r)$, $g_H = h - t_H$ and $\Delta_T = r/2 - p + 3p^2/(2r)$. For $r < p < h$, $t_0 = t_L = g_L = 0$, $t_H = p$, $g_H = h - p$ and $\Delta_T = p$.

Proposition A.9 (Reserves)**.** *Fix a reserve, and allow class values below it. Keep $G_H(y) \le G_L(y)$ for every y, the costs, the slot and a strengthening $F_1 \le F_0$, and compute each class object with the reserve. Suppose $k < \rho\kappa \min\{\Delta_0, \Delta_1\}$.*

(a) If $B_0(\pi) < c_H < B_1(M)$, entry is the same at the two strengths: $\mathsf{E}_0 = \mathsf{E}_1 = \rho + (1-\rho)(b + \pi\lambda)$.

(b) If $B_0(\pi) < c_H < B_0(M)$ and $B_1(M) < c_H$, entry falls strictly: $\mathsf{E}_0 = \rho + (1-\rho)(b+\pi\lambda) > \rho = \mathsf{E}_1$.

Each economy has a unique outcome under every pair of tie rules.

Proof of Proposition A.9. Orders. The zero-cost rival prepares after every price, so entry is at least ρ. Lemma A.3 uses only (E1), (E4), (E6), $e \in [\rho, 1]$ and $\Delta_T \ge 0$, so it holds with the reserve kernel. Every correct order earns at least $\rho\kappa\Delta_i - k > 0$ under every candidate, and every wrong-signed order at most $-k$. Full orders are forced at both strengths, and the posteriors are (m, π, M).

Entry. Gross profit falls with incumbent strength at every belief, because $(\theta - \max\{p, R\})_+$ is nonincreasing in R. In part (a), $B_1(\pi) \le B_0(\pi) < c_H < B_1(M) \le B_0(M)$, so at both strengths only the order 1 recruits, and $B_i(m) \le B_i(\pi)$ keeps the sale below the cost. In part (b), the weak strength has the same pattern, and at the strong strength $B_1(\mu) \le B_1(M) < c_H$ at every posterior. Every realized comparison is strict.

The tuples. At each strength the tuple has full orders, posteriors (m, π, M), entry $(\rho, \rho, 1)$ or (ρ, ρ, ρ) as above, $\zeta(P(x)) = [e(x) - \rho]/(1-\rho)$, prices $P_i(x) = t_{0,i} + e_i(x)\, w_i(\mu(x))$, $\bar{\mu}(P_i(x)) = \mu(x)$, and truthful bids. Since $\Delta_i > 0$ and entry is nondecreasing in the posterior, the three prices differ, so (E2) holds. (E1), (E3) and (E4) hold by the computations above, (E5) by the strict bounds, and (E6) by truthful bids. The bounds hold for every candidate, so each tuple is unique, under every pair of tie rules. A purchase has probability $b + \pi\lambda$, which gives both entry formulas. □

Reserves in the example. At each reserve and strength below, three margins decide the outcome: the trading margin $\rho\kappa\Delta_T - k$, the prior-cost margin $c_H - B(1/2)$ and the top-posterior margin $B(M) - c_H$. Each follows from the kernel formulas above, and the triples list them in this order.

- *A reserve that excludes the low class.* At the reserve $101/100$ with point values, the low class lies below the reserve and the high class above both incumbent supports, so the formulas above apply. The margins are $(0.038167, 1.512521, 0.561220)$ at r_0 and $(0.047239, 1.835008, 0.089706)$ at r_1. Part (a) applies, and entry is 0.522757 at both strengths. Proceeds exceed their levels at the reserve $1/2$ by 0.047856 and 0.121853. For the narrow bands at $11/10$, the margins are $(0.048076, 1.552083, 0.503375)$ and $(0.047572, 1.850833, 0.066568)$, with proceeds gains 0.027458 and 0.150186.

- *Deterrence.* At the reserve $31/20$, the reserve exceeds r_0, so the weak incumbent never meets it, and at r_0 the objects are $t_0 = t_L = g_L = 0$, $t_H = p$ and $g_H = h - p$. The margins are $(0.084215, 1.775000, 0.177445)$ at r_0 and $(0.057405, 1.950208, -0.078730)$ at r_1. Part (b) applies: entry is 0.522757 at r_0 and 0.250000 at r_1. Proceeds exceed their levels at the reserve $1/2$ by 0.110114 and 0.020542. The point values and the narrow bands give the same objects here, because the reserve lies above the low band and below the high band. The strong upper profit equals c_H at $p_H = \sqrt{2r_1(h - c_H/M) - r_1^2} \approx 1.325270$, and the weak upper profit equals c_H at $h - c_H/M \approx 1.792723$; the reserve lies inside this interval, at distances 0.224730 and 0.242723 from its ends, and $p_H - r_0 = 0.125270$.

- *Why deterrence excludes the weak incumbent.* For point values and $1 < p \le r_0$, the formulas above give $\Delta_T(r_0) - 17/20 = (p-1)(5p+1)/4 \ge 0$ and $\Delta_T(r_1) - 1 = (p-1)^2/2 \ge 0$, so the trading margins stay at least 0.037150 and 0.047235. The strong upper profit is $M(h - r_1/2 - p^2/(2r_1))$, which at $p = r_0$ exceeds c_H by 0.038544 and is larger at every lower reserve. The weak upper profit is larger still, and both prior profits lie below c_H. So both strengths recruit after a purchase only, and entry is flat. For the narrow bands with $21/20 < p \le r_0$, the low band lies below p and the high band above r_1, so every band object equals the point-value object, and the same conclusion holds. Strict deterrence by a reserve that excludes the low class therefore needs $p > r_0$.

- *Boundary reserves.* At $p = p_H$, the purchase leaves the high-cost rival indifferent at r_1: when an indifferent high-cost rival prepares, entry is $(\rho, \rho, 1)$, and when it may prepare with any probability, the last component takes any value in $[\rho, 1]$; the trading margin there is 0.050792. At $p = h$ with point values, every acquisition profit is zero, the definition fixes zero-cost preparation, $\Delta_T = h$, the bound $h\rho\kappa > k$ forces full orders, and entry is 0.250000 with proceeds 1.250000.

A.8 Empirical implications

This subsection states and proves the results that Section 8 summarizes. The incumbent distribution F is any continuous distribution, the class distributions satisfy the class ranking (1), and the prior $\pi \in (0, 1)$ is arbitrary; as in Lemma A.2, $\Delta_T > 0$ and $\gamma = g_H - g_L > 0$. Each result names the trading pattern and the tie rules it uses. Every statement concerns one target in one public state, with its prior, slot and incumbent distribution fixed, and every probability, mean and covariance is taken over the order opportunity of that state. Write $M = M(\pi)$ and $m = m(\pi)$; $\bar{f}(x) = \Pr(X = x)$ for the frequency of the order x per order opportunity; $v = \Pr(X \ne 0)$ for the volume; $w_F(\mu) = t_L - t_0 + \Delta_T\mu$ for the entry gain at the belief μ against F (Appendix A.2); and $\chi_F = \mathbf{1}\{B_F(M) \ge c_H\}$. The entry

gain rises in μ with slope Δ_T and is positive at every interior belief. By Lemma A.2(d), entry after an order is ρ when its posterior lies below τ and one when it lies above; at the posterior τ it is one when an indifferent high-cost rival prepares, and any value in $[\rho, 1]$ when an indifferent high-cost rival may prepare with any probability.

The purchase impact is $I_+ = P(1) - P(0)$, the sale impact is $I_- = P(0) - P(-1)$, and the price gap $P(1) - P(-1) = I_+ + I_-$ is the difference between the conditional execution prices of a unit purchase and a unit sale. The model has no separate quote stage: each price is the competitive value conditional on the order. The weighted impact difference is $\bar{f}(1)I_+ - \bar{f}(-1)I_-$.

Under full two-sided trade, $\sigma_H = \delta_1$ and $\sigma_L = \delta_{-1}$, Lemma A.1(d) gives the posteriors (m, π, M) after the orders $(-1, 0, 1)$, and

$$\begin{gathered} \bar{f}(1) = b + \pi\lambda, \qquad \bar{f}(0) = 1 - 2b - \lambda, \qquad \bar{f}(-1) = b + (1-\pi)\lambda, \\ v = 2b + \lambda, \qquad \mathbb{E}[X] = \lambda(2\pi - 1). \end{gathered} \tag{A.20}$$

Subtracting the prior from each posterior gives

$$\bar{f}(1)\,(M - \pi) = \bar{f}(-1)\,(\pi - m) = \pi(1-\pi)\lambda. \tag{A.21}$$

The informed share of nonzero orders is $\vartheta = \lambda/v = u/(2+u)$ at every prior. The posterior gap is $M - m = \pi(1-\pi)\lambda v/[\bar{f}(1)\bar{f}(-1)] \le \vartheta$, with equality exactly at the prior $1/2$, where $M + m = 1$ and $\vartheta = 2M - 1 = M - m$. When $\pi < \tau$, entry after a sale and after no order is ρ under both tie rules for the rival.

Proof of the full-trade identities. Under full two-sided trade, class H places the orders $(-1, 0, 1)$ with probabilities $(b,\ 1-2b-\lambda,\ b+\lambda)$ and class L with probabilities $(b+\lambda,\ 1-2b-\lambda,\ b)$. Averaging with the weights π and $1-\pi$ gives (A.20), and Lemma A.1(d) gives the posteriors. Bayes' rule gives $\bar{f}(1)\,M = \pi(b+\lambda)$ and $\bar{f}(-1)\,m = \pi b$. Subtracting $\pi\bar{f}(1)$ from the first and $\pi\bar{f}(-1)$ from the second gives $\bar{f}(1)(M-\pi) = \pi[b+\lambda-\bar{f}(1)] = \pi(1-\pi)\lambda$ and $\bar{f}(-1)(\pi-m) = \pi[\bar{f}(-1)-b] = \pi(1-\pi)\lambda$, which is (A.21). Adding $M - \pi$ and $\pi - m$, with $\bar{f}(1) + \bar{f}(-1) = v$, gives the posterior gap. With $u = \lambda/b$, $\lambda/v = \lambda/(2b+\lambda) = u/(2+u)$. The posterior gap is ϑ times $\pi(1-\pi)v^2/[\bar{f}(1)\bar{f}(-1)]$, and

$$\bar{f}(1)\,\bar{f}(-1) - \pi(1-\pi)v^2 = b^2 + b\lambda + \pi(1-\pi)\lambda^2 - \pi(1-\pi)(2b+\lambda)^2 = b(b+\lambda)(1-2\pi)^2 \ge 0,$$

with equality exactly at the prior $1/2$, because $b > 0$. At the prior $1/2$, $M = (b+\lambda)/v$ and $m = b/v$, so $M + m = 1$ and $2M - 1 = \lambda/v$. The entry gain is $w_F(\mu) = w_L + \Delta_T\mu \ge \Delta_T\mu$, since $w_L \ge 0$ (Appendix A.2), and is therefore positive at every $\mu > 0$. When $\pi < \tau$, the posteriors after a sale and after no order are $m < \pi < \tau$, and Lemma A.2(d) gives entry ρ after both, with no posterior at τ.

These steps use parts (a), (b) and (d) of Lemma A.1, which hold for every $\beta < 1/2$. □

Proposition A.10 (The price gap and the weighted test)**.**

(i) Every trading pattern. *In every equilibrium,*

$$\bar{f}(1)I_+ - \bar{f}(-1)I_- = \mathbb{E}[P(X)] - P(0) = e(0)\,\Delta_T\,[\pi - \mu(0)] + \sum_x \bar{f}(x)\,[e(x) - e(0)]\,w_F(\mu(x)).$$

(ii) The weighted test. *Under full two-sided trade,*

$$\bar{f}(1)I_+ - \bar{f}(-1)I_- = \bar{f}(1)\,[e(1) - e(0)]\,w_F(M) - \bar{f}(-1)\,[e(0) - e(-1)]\,w_F(m).$$

If also $\pi < \tau$, then

$$I_+ - \frac{\bar{f}(-1)}{\bar{f}(1)}\,I_- = [e(1) - \rho]\,w_F(M),$$

which is positive exactly when a purchase recruits, that is, exactly when $e(1) > \rho$, under both tie rules for the rival.

(iii) Signs. *Let full two-sided trade hold, and let an indifferent high-cost rival prepare. Entry after the orders $(-1, 0, 1)$ and the weighted impact difference are:*

Preparation threshold	*Entry after* $(-1, 0, 1)$	$\bar{f}(1)I_+ - \bar{f}(-1)I_-$
$\tau \le m$	$(1, 1, 1)$	0
$m < \tau \le \pi$	$(\rho, 1, 1)$	$-(1-\rho)\,\bar{f}(-1)\,w_F(m) < 0$
$\pi < \tau \le M$	$(\rho, \rho, 1)$	$(1-\rho)\,\bar{f}(1)\,w_F(M) > 0$
$M < \tau$	(ρ, ρ, ρ)	0

When an indifferent high-cost rival may prepare with any probability, the table holds for $\tau \notin \{m, \pi, M\}$. At $\tau = M$ the weighted impact difference is $[e(1) - \rho]\,\bar{f}(1)\,w_F(M) \ge 0$, positive exactly when $e(1) > \rho$; at $\tau = m$ it is $-[1 - e(-1)]\,\bar{f}(-1)\,w_F(m) \le 0$, negative exactly when $e(-1) < 1$. At $\tau = \pi$, full two-sided trade is an equilibrium with one entry $e(0) \in [\rho, 1]$ after no order exactly when it is an equilibrium with every such entry, and as $e(0)$ runs over $[\rho, 1]$ the weighted impact difference $\bar{f}(1)\,[1 - e(0)]\,w_F(M) - \bar{f}(-1)\,[e(0) - \rho]\,w_F(m)$ takes every value from $-(1-\rho)\,\bar{f}(-1)\,w_F(m)$ to $(1-\rho)\,\bar{f}(1)\,w_F(M)$.

(iv) The price gap. *Under full two-sided trade with $\pi < \tau$,*

$$P(1) - P(-1) = \rho\Delta_T\,(M - m) + [e(1) - \rho]\,w_F(M), \qquad \rho\Delta_T\,(M - m) = \frac{v}{\bar{f}(1)}\,I_-,$$

a class component and an entry component. If the trading margin $\rho\kappa(\pi)\Delta_T - k$ is positive, with $\kappa(\pi) = \min\{m, 1-M\}$, every equilibrium has full two-sided trade, under both tie rules for the informed trader, whether or not the rival sees the price. If also $\pi < \tau$ and the rival does not see the price, its belief stays at π, entry is ρ after every order, the price gap is the class component $\rho\Delta_T\,(M-m)$, and the weighted impact difference is zero.

(v) The unweighted comparison. *Under full two-sided trade with $\pi < \tau$,*

$$I_+ - I_- = \rho\Delta_T \,\frac{\pi(1-\pi)\lambda^2(1-2\pi)}{\bar{f}(1)\,\bar{f}(-1)} + [e(1)-\rho]\,w_F(M).$$

If $\pi \geq 1/2$, $\rho \leq 1/2$ and an indifferent high-cost rival prepares, then under full two-sided trade $I_+ > I_-$ exactly when $\pi < \tau \leq M$, that is, exactly when a purchase raises entry above its level after no order.

(vi) Strength. *Let F_r be a path along which Δ_T rises strictly, hold the prior and the slot fixed, and write $w_r = w_{F_r}$ and $\chi_r = \chi_{F_r}$.*

(a) For every belief μ,

$$w_r(\mu) = pF_r(p) + \int_p^\infty \big[1-(1-\mu)G_L(y)-\mu G_H(y)\big]\big[1-F_r(y)\big]\,dy\,:$$

the value of rescuing the sale when the incumbent misses the reserve, plus the value of competition above the reserve. From r to $r' > r$, the first term falls by $p\,[F_r(p)-F_{r'}(p)] \geq 0$, and the second rises by $\int_p^\infty [1-(1-\mu)G_L(y)-\mu G_H(y)]\,[F_r(y)-F_{r'}(y)]\,dy \geq \mu\,[\Delta_T(r')-\Delta_T(r)]$. The entry gain $w_r(\mu)$ rises from r to r' exactly when the rise of the second term exceeds the fall of the first.

(b) If the probability $F_r(p)$ that the incumbent misses the reserve is the same at every strength of the path, $w_r(\mu)$ rises strictly in r for every $\mu \in (0,1]$. If F_r is uniform on $[0,r]$ with $r > p$, then

$$\frac{d}{dr}\,w_r(\mu) = \frac{\mathbb{E}_\mu\big[\min\{\theta,r\}^2\big]-3p^2}{2r^2},$$

where $\mathbb{E}_\mu$ takes θ from $(1-\mu)G_L+\mu G_H$. So $w_r(\mu)$ rises strictly in r for every μ when $p < \min\{\underline{\theta},\underline{r}\}/\sqrt{3}$, where $\underline{\theta}$ is the lowest value of the rival and $\underline{r}$ the lowest strength on the path.

(c) Let an indifferent high-cost rival prepare, and let full two-sided trade hold at every strength of an interval on which $\pi < \tau(r)$ and $w_r(M)$ rises strictly. Where $\chi_r = 1$, both components of the price gap, both impacts, the price gap and the weighted impact

difference rise strictly with r. Where $\chi_r = 0$, the entry component and the weighted impact difference are zero, and the class component, both impacts and the price gap rise strictly with r.

(d) The indicator χ_r is nonincreasing in r. Let also $F_r(y)$ be continuous in r at every y; then the strengths with $\chi_r = 1$ form a closed subset of the path. Let r_C be a strength with $\chi_{r_C} = 1$ and $\chi_r = 0$ for every $r > r_C$, and let full two-sided trade and $\pi < \tau(r)$ hold on a neighborhood of r_C. When an indifferent high-cost rival prepares, $P(1)$, I_+ and the price gap are continuous from the left at r_C, and their right limits lie below their values at r_C by $(1-\rho)\, w_{r_C}(M) > 0$; the weighted impact difference falls by $(1-\rho)\, \bar{f}(1)\, w_{r_C}(M)$ in the same way; $P(-1)$, $P(0)$ and I_- are continuous at r_C. When an indifferent informed trader may mix over its best orders and an indifferent high-cost rival may prepare with any probability, entry after a purchase at r_C can take any value in $[\rho, 1]$, and the price gap at r_C any value between its two one-sided limits.

Proof of Proposition A.10. By Lemma A.2(b), $P(x) = t_0 + e(x)\, w_F(\mu(x))$ after every order, in every equilibrium and under both tie rules for the rival.

(i) Since $P(1) - P(0) = I_+$ and $P(-1) - P(0) = -I_-$, $\mathbb{E}[P(X)] - P(0) = \sum_x \bar{f}(x)\, [P(x) - P(0)] = \bar{f}(1) I_+ - \bar{f}(-1) I_-$. Every order has positive probability by Lemma A.1(a), so Bayes' rule gives $\sum_x \bar{f}(x)\, \mu(x) = \pi$, and since w_F is affine, $\sum_x \bar{f}(x)\, w_F(\mu(x)) = w_F(\pi)$. Hence

$$\begin{aligned}\mathbb{E}[P(X)] - P(0) &= \sum_x \bar{f}(x)\, e(x)\, w_F(\mu(x)) - e(0)\, w_F(\mu(0)) \\ &= \sum_x \bar{f}(x)\, [e(x) - e(0)]\, w_F(\mu(x)) + e(0)\, [w_F(\pi) - w_F(\mu(0))],\end{aligned}$$

and $w_F(\pi) - w_F(\mu(0)) = \Delta_T\, [\pi - \mu(0)]$.

(ii) Under full two-sided trade $\mu(0) = \pi$, so the first term of (i) is zero, and the posteriors after the orders 1 and -1 are M and m. The sum in (i) then has the two terms of the display. If $\pi < \tau$, entry after a sale and after no order is ρ by the standing paragraph, so the display equals $\bar{f}(1)\, [e(1) - \rho]\, w_F(M)$; dividing by $\bar{f}(1) > 0$ gives the second display. Since $w_F(M) > 0$ and $e(1) \geq \rho$, it is positive exactly when $e(1) > \rho$. No step uses the rival's rule at a posterior equal to τ.

(iii) Under full two-sided trade $m < \pi < M$. When an indifferent high-cost rival prepares, Lemma A.2(d) gives $e(x) = 1$ when $\mu(x) \geq \tau$ and $e(x) = \rho$ otherwise, which is the entry column. The display of (ii) gives the last column, and $w_F(m), w_F(M) > 0$ give the signs. When an indifferent high-cost rival may prepare with any probability, Lemma A.2(d) differs

only after an order whose posterior equals τ, so the table holds for $\tau \notin \{m, \pi, M\}$. At $\tau = M$, $e(-1) = e(0) = \rho$ and $e(1) \in [\rho, 1]$; at $\tau = m$, $e(0) = e(1) = 1$ and $e(-1) \in [\rho, 1]$; the display of (ii) gives the two values. At $\tau = \pi$, $e(-1) = \rho$, $e(1) = 1$, and $e(0) \in [\rho, 1]$ is the rival's choice at the posterior τ. The posteriors do not depend on entry, and by Lemma A.3(d) entry after the zero order enters no payoff of the informed trader, so the informed trader's conditions are the same for every $e(0)$. Entry is nondecreasing in the posterior for every $e(0) \in [\rho, 1]$, so the paragraph on constructed tuples in Appendix A.2 supplies the prices, the preparation rule and the price beliefs, and (E3) holds at the posterior τ. Full two-sided trade is therefore an equilibrium with one such $e(0)$ exactly when it is an equilibrium with every such $e(0)$. By (ii), the weighted impact difference is $\bar{f}(1)\,[1 - e(0)]\,w_F(M) - \bar{f}(-1)\,[e(0) - \rho]\,w_F(m)$, which is affine and strictly decreasing in $e(0)$, with the value $(1 - \rho)\,\bar{f}(1)\,w_F(M)$ at $e(0) = \rho$ and $-(1 - \rho)\,\bar{f}(-1)\,w_F(m)$ at $e(0) = 1$.

(iv) With $\pi < \tau$, $e(-1) = e(0) = \rho$, so

$$P(1) - P(-1) = e(1)\,w_F(M) - \rho\,w_F(m) = \rho\Delta_T\,(M - m) + [e(1) - \rho]\,w_F(M).$$

Also $I_- = \rho\,[w_F(\pi) - w_F(m)] = \rho\Delta_T\,(\pi - m)$. By (A.21), $\pi - m = \pi(1 - \pi)\lambda/\bar{f}(-1)$, and by the standing paragraph $M - m = \pi(1 - \pi)\lambda v/[\bar{f}(1)\bar{f}(-1)]$, so $M - m = (\pi - m)\,v/\bar{f}(1)$, which gives the second identity.

In the economy where the rival sees the price, Lemma A.3(c) bounds K_H and K_L below by $\rho\kappa(\pi)\Delta_T$ in every candidate. Where the rival does not see the price, its belief is π after every order, so it prepares with one probability $e \in [\rho, 1]$ after every order. Preparation is then independent of the class and of the order, the competitive price is $P(x) = t_0 + e\,w_F(\mu(x))$ by the argument of Lemma A.2(b), and the payoffs of Lemma A.3 hold with $e(x) = e$. Its bounds in part (c) use only $e \geq \rho$ and the posterior bounds of Lemma A.1(b), so they hold as well. In both economies, $\rho\kappa(\pi)\Delta_T > k$ gives $K_H > k$ and $K_L > k$ in every candidate, and Lemma A.3(a) and (b) make class H buy and class L sell with probability one, with no indifference, under both tie rules for the informed trader.

Let also $\pi < \tau$, with a hidden price. Gross profit at the belief π is $B_F(\pi) = c_H + \gamma(\pi - \tau) < c_H$, so the high-cost rival abstains, and entry is ρ after every order. The orders are full, the posteriors are (m, π, M), and $P(x) = t_0 + \rho\,w_F(\mu(x))$. The price gap is $\rho\,[w_F(M) - w_F(m)] = \rho\Delta_T\,(M - m)$. The computation of (i) and (ii) uses only this price formula, Bayes' rule and full two-sided trade, so with $e \equiv \rho$ it gives a zero weighted impact difference.

(v) With $\pi < \tau$, $I_+ = e(1)\,w_F(M) - \rho\,w_F(\pi) = \rho\Delta_T\,(M - \pi) + [e(1) - \rho]\,w_F(M)$ and $I_- =$

$\rho\Delta_T\,(\pi - m)$. By (A.21),

$$(M - \pi) - (\pi - m) = \pi(1-\pi)\lambda\,\frac{\bar f(-1) - \bar f(1)}{\bar f(1)\,\bar f(-1)},$$

and $\bar f(-1) - \bar f(1) = \lambda(1 - 2\pi)$ by (A.20). This gives the formula.

Let $\pi \geq 1/2$, $\rho \leq 1/2$, and let an indifferent high-cost rival prepare. Then $\bar f(1) \geq \bar f(-1)$, and (A.21) gives $M - \pi \leq \pi - m$. Take the rows of the table in (iii).

- $\pi < \tau \leq M$, entry $(\rho, \rho, 1)$. The formula with $e(1) = 1$ and $w_F(M) = w_L + \Delta_T M$ gives $I_+ - I_- = (1-\rho)\,w_L + \Delta_T\,(M - 2\rho\pi + \rho m)$. Since $\rho \leq 1/2$, $M - 2\rho\pi \geq M - \pi > 0$, and $w_L \geq 0$, so $I_+ > I_-$.
- $M < \tau$, entry (ρ, ρ, ρ). The formula gives $I_+ - I_- = \rho\Delta_T\,[(M - \pi) - (\pi - m)] \leq 0$.
- $\tau \leq m$, entry $(1, 1, 1)$. Then $I_+ = \Delta_T\,(M - \pi)$ and $I_- = \Delta_T\,(\pi - m)$, so $I_+ \leq I_-$.
- $m < \tau \leq \pi$, entry $(\rho, 1, 1)$. Then $I_+ = \Delta_T\,(M - \pi)$ and $I_- = w_F(\pi) - \rho\,w_F(m) = \rho\Delta_T\,(\pi - m) + (1-\rho)\,w_F(\pi)$. Hence $I_+ - I_- \leq (1-\rho)\,\Delta_T\,(\pi - m) - (1-\rho)\,w_F(\pi) < 0$, because $w_F(\pi) \geq \Delta_T\pi > \Delta_T\,(\pi - m)$.

So $I_+ > I_-$ exactly in the row $\pi < \tau \leq M$, the only row in which $e(1) > e(0)$.

(vi) The prior and the slot are fixed, so M, m, $\bar f(x)$ and v do not depend on r.

(a) Under the class ranking $G_j(p^-) = 0$, and Appendix A.1 gives $w_j = t_j - t_0 = pF(p) + \int_p^\infty [1 - G_j(y)][1 - F(y)]\,dy$. Averaging with the weights $1 - \mu$ and μ gives the display. Along the path $F_{r'} \leq F_r$, which gives the change of the first term. The integrand of the change of the second term carries $1 - (1-\mu)G_L - \mu G_H = 1 - G_L + \mu D \geq \mu D$, and Lemma A.5 gives $\int_p^\infty [F_r(y) - F_{r'}(y)]\,D(y)\,dy = \Delta_T(r') - \Delta_T(r)$. The last sentence compares the two changes.

(b) If $F_r(p)$ does not depend on r, the first term does not change, and by (a) the second rises by at least $\mu\,[\Delta_T(r') - \Delta_T(r)] > 0$ for $\mu > 0$. Let F_r be uniform on $[0, r]$ with $r > p$, and write $\bar G_\mu(y) = 1 - (1-\mu)G_L(y) - \mu G_H(y)$, the probability that $\theta > y$ under $(1-\mu)G_L + \mu G_H$. Then

$$w_r(\mu) = \frac{p^2}{r} + \frac{1}{r}\int_p^r (r - y)\,\bar G_\mu(y)\,dy.$$

The function $r \mapsto \int_p^r (r-y)\bar G_\mu(y)\,dy = \int_p^r \int_p^z \bar G_\mu(y)\,dy\,dz$ has the continuous derivative

$\int_p^r \bar{G}_\mu(y)\,dy$, so $w_r(\mu)$ is differentiable in r, with

$$\frac{d}{dr}\,w_r(\mu) = -\frac{p^2}{r^2} + \frac{r\int_p^r \bar{G}_\mu(y)\,dy - \int_p^r (r-y)\bar{G}_\mu(y)\,dy}{r^2} = \frac{\int_p^r y\bar{G}_\mu(y)\,dy - p^2}{r^2}.$$

Every value of the rival is at least p, so $\bar{G}_\mu = 1$ below p, and $\mathbb{E}_\mu[\min\{\theta, r\}^2] = \int_0^r 2y\,\bar{G}_\mu(y)\,dy = p^2 + 2\int_p^r y\bar{G}_\mu(y)\,dy$. This gives the formula. On the path $r \geq \underline{r}$, and every value satisfies $\theta \geq \underline{\theta}$, so $\min\{\theta, r\} \geq \min\{\underline{\theta}, \underline{r}\} > \sqrt{3}\,p$, and the derivative is positive.

(c) Under the preparing rule, Lemma A.2(d) and $\pi < \tau(r)$ give entry $(\rho, \rho, \rho + (1-\rho)\chi_r)$ after the orders $(-1, 0, 1)$, because $B_r(M) \geq c_H$ exactly when $M \geq \tau(r)$. By (iv) and (v), the class component $\rho\Delta_T(r)(M - m)$ and the sale impact $\rho\Delta_T(r)(\pi - m)$ rise strictly with Δ_T. The entry component is $(1-\rho)\chi_r\,w_r(M)$, the purchase impact is $\rho\Delta_T(r)(M - \pi) + (1-\rho)\chi_r\,w_r(M)$, and by (ii) the weighted impact difference is $(1-\rho)\chi_r\,\bar{f}(1)\,w_r(M)$. Where $\chi_r = 1$ each of these rises strictly, because $w_r(M)$ does; where $\chi_r = 0$ the entry component and the weighted impact difference are zero, and the class component, both impacts and the price gap rise strictly with Δ_T.

(d) Each g_j is nonincreasing along the path (Lemma A.5), so $B_r(M) = (1-M)g_L(F_r) + Mg_H(F_r)$ is nonincreasing, and χ_r is nonincreasing. In $g_j(F_r) = \int_p^\infty F_r(y)[1 - G_j(y)]\,dy$ the integrand is bounded by $1 - G_j$, which is integrable because G_j has a finite mean; when $F_r(y)$ is continuous in r, dominated convergence makes $B_r(M)$ continuous in r, and $\{r : B_r(M) \geq c_H\}$ is closed in the path. In the same way, the display of (a) makes $w_r(\mu)$ continuous in r, and $t_0 = p[1 - F_r(p)]$ is continuous in r. At r_C, continuity and $B_r(M) < c_H$ for $r > r_C$ give $B_{r_C}(M) = c_H$, that is $M = \tau(r_C)$.

On the neighborhood of r_C, $P(-1) = t_0 + \rho\,w_r(m)$ and $P(0) = t_0 + \rho\,w_r(\pi)$ are continuous in r, and so is I_-. When an indifferent high-cost rival prepares, $e(1) = 1$ for $r \leq r_C$ and $e(1) = \rho$ for $r > r_C$, so $P(1) = t_0 + e(1)\,w_r(M)$ is continuous from the left at r_C and its right limit is lower by $(1-\rho)\,w_{r_C}(M) > 0$; I_+ and the price gap move with $P(1)$, and by (ii) the weighted impact difference $\bar{f}(1)\,[e(1) - \rho]\,w_r(M)$ falls by $(1-\rho)\,\bar{f}(1)\,w_{r_C}(M)$.

When an indifferent high-cost rival may prepare with any probability, the posterior after a purchase at r_C equals $\tau(r_C)$, so (E3) allows every $e(1) \in [\rho, 1]$. The class L conditions and the posteriors do not depend on $e(1)$. On the right of r_C, full two-sided trade has entry ρ after a purchase, so the purchase margin $\rho\Delta_T(r)(1 - M) - k$ is nonnegative there; its limit $\rho\Delta_T(r_C)(1 - M) - k$ is nonnegative as well. At r_C the purchase earns $e(1)\Delta_T(r_C)(1 - M) - k$, which is positive for every $e(1) \in (\rho, 1]$ and nonnegative at $e(1) = \rho$, where an indifferent informed trader who may mix buys with probability one.

The paragraph on constructed tuples in Appendix A.2 completes each tuple. The price gap at r_C is $\rho\Delta_T(r_C)(M-m) + [e(1)-\rho]\, w_{r_C}(M)$, affine in $e(1)$, and it takes every value between its two one-sided limits. □

Proposition A.11 (What order counts and prices identify)**.** *Let full two-sided trade hold.*

(a) A known prior other than $1/2$. *If $\pi \neq 1/2$ is known, the counts identify the slot:*

$$\lambda = \frac{\mathbb{E}[X]}{2\pi - 1}, \qquad b = \frac{v-\lambda}{2}, \qquad \beta = \frac{v-\lambda}{2(1-\lambda)}, \qquad \vartheta = \frac{\lambda}{v}.$$

Over independent order opportunities, the mean order divided by $2\pi - 1$ estimates λ without bias, with variance $\big(v - \mathbb{E}[X]^2\big)/(2\pi-1)^2$ divided by the number of opportunities. If also $\pi < \tau$, the sale impact identifies $\rho\Delta_T = \bar{f}(-1)\, I_-/[\pi(1-\pi)\lambda]$.

(b) The prior $1/2$. *At the prior $1/2$ the order distribution is $(v/2, 1-v, v/2)$, so the counts identify v and no other function of the slot. A share $\vartheta' \in (0,1)$ belongs to exactly one pair (λ', β') with the volume v, namely $\lambda' = v\vartheta'$ and $\beta' = v(1-\vartheta')/[2(1-v\vartheta')]$, and this pair is a slot of the model, $\beta' \leq 1/3$, exactly when $\vartheta' \geq 3 - 2/v$. Let $\tau > 1/2$. Under full two-sided trade at this slot, entry is $\rho + (1-\rho)(v/2)\,\mathbf{1}\{(1+\vartheta')/2 \geq \tau\}$ when an indifferent high-cost rival prepares. Full two-sided trade at this slot is an equilibrium exactly when $\vartheta' \leq 1 - 2k/(\rho\Delta_T)$ if an indifferent informed trader may mix over its best orders, and exactly when $\vartheta' < 1 - 2k/(\rho\Delta_T)$ if an indifferent informed trader abstains.*

(c) The sale price. *If entry after a sale and after no order is ρ and $\rho\Delta_T$ is known, the sale impact gives $\pi(1-\pi)\lambda = \bar{f}(-1)\, I_-/(\rho\Delta_T)$, and with the counts*

$$\lambda = \frac{2\bar{f}(-1)\, I_-}{\rho\Delta_T} + \sqrt{\Big(\frac{2\bar{f}(-1)\, I_-}{\rho\Delta_T}\Big)^2 + \mathbb{E}[X]^2}, \qquad \pi = \frac{1}{2}\Big(1 + \frac{\mathbb{E}[X]}{\lambda}\Big),$$

and then b and β as in part (a). At the prior $1/2$, $\lambda = 4\bar{f}(-1)\, I_-/(\rho\Delta_T)$ and $\vartheta = 2I_-/(\rho\Delta_T)$.

Proof of Proposition A.11. Let full two-sided trade hold. The counts are the order distribution $\bar{f}$.

(a) By (A.20), $v = 2b + \lambda$ and $\mathbb{E}[X] = \lambda(2\pi - 1)$. With $\pi \neq 1/2$ known, these give λ, then $b = (v-\lambda)/2$, $\beta = b/(1-\lambda)$ and $\vartheta = \lambda/v$. The order X takes values in $\{-1, 0, 1\}$, so $X^2 = \mathbf{1}\{X \neq 0\}$, $\mathbb{E}[X^2] = v$ and the variance of X is $v - \mathbb{E}[X]^2$. The mean of a sample of independent orders, divided by $2\pi - 1$, has mean λ and variance $(v - \mathbb{E}[X]^2)/(2\pi-1)^2$ divided by the sample size. If also $\pi < \tau$, entry after a sale and after no order is ρ, so $I_- = \rho\,[w_F(\pi) - w_F(m)] = \rho\Delta_T\,(\pi - m)$, and (A.21) gives $I_- = \rho\Delta_T\,\pi(1-\pi)\lambda/\bar{f}(-1)$. Solving for $\rho\Delta_T$ gives the formula.

(b) At the prior $1/2$, (A.20) gives $\bar{f}(\pm 1) = b + \lambda/2 = v/2$ and $\bar{f}(0) = 1 - v$, so the order distribution is a function of v alone. Given v and ϑ', the equations $\lambda' = v\vartheta'$ and $2b' + \lambda' = v$ give $b' = v(1-\vartheta')/2$ and $\beta' = b'/(1-\lambda') = v(1-\vartheta')/[2(1-v\vartheta')]$, a unique pair. Since $v = 1 - \bar{f}(0) < 1$, $0 < \lambda' < 1$ and $\beta' > 0$. The inequality $\beta' \le 1/3$ is $3v(1-\vartheta') \le 2(1-v\vartheta')$, that is $v\vartheta' \ge 3v - 2$, that is $\vartheta' \ge 3 - 2/v$.

At this slot and the prior $1/2$, $M' = (1+\vartheta')/2$ and $m' = (1-\vartheta')/2$ by the standing paragraph. Let $\tau > 1/2$. Under full two-sided trade, the posteriors after a sale and after no order are $m' < 1/2 < \tau$, so entry after them is ρ, and when an indifferent high-cost rival prepares, entry after a purchase is one exactly when $M' \ge \tau$ and ρ otherwise. Weighting by $\bar{f}(1) = v/2$ gives the entry formula. By Lemma A.3, the sale earns $K_L - k = \rho\Delta_T\, m' - k$, and the purchase earns $K_H - k = e(1)\Delta_T\,(1 - M') - k \ge \rho\Delta_T\, m' - k$, because $1 - M' = m'$ and $e(1) \ge \rho$. Wrong-signed orders never pay. When an indifferent informed trader may mix over its best orders, full orders are optimal exactly when $K_L \ge k$ and $K_H \ge k$, that is exactly when $\rho\Delta_T(1-\vartheta')/2 \ge k$; when an indifferent informed trader abstains, the two inequalities are strict. The first condition is $\vartheta' \le 1 - 2k/(\rho\Delta_T)$, and the second the same with strict inequality. The paragraph on constructed tuples in Appendix A.2 completes the tuple under either tie rule for the rival, so these are also the conditions for an equilibrium.

(c) With entry ρ after a sale and after no order, $I_- = \rho\Delta_T\,(\pi - m)$, and (A.21) gives $q := \bar{f}(-1)\, I_-/(\rho\Delta_T) = \pi(1-\pi)\lambda$. Since $(2\pi - 1)^2 = 1 - 4\pi(1-\pi)$,

$$\mathbb{E}[X]^2 = \lambda^2(2\pi-1)^2 = \lambda^2 - 4q\lambda, \qquad \text{so} \qquad \lambda^2 - 4q\lambda - \mathbb{E}[X]^2 = 0.$$

Since $q > 0$, the product of the two roots is $-\mathbb{E}[X]^2 \le 0$ and their sum $4q$ is positive, so exactly one root is positive, $\lambda = 2q + \sqrt{4q^2 + \mathbb{E}[X]^2}$, which is the display. Then $2\pi - 1 = \mathbb{E}[X]/\lambda$, and part (a) gives b and β from v and λ. At the prior $1/2$, $\mathbb{E}[X] = 0$, so $\lambda = 4q$, and $\vartheta = \lambda/v = 2I_-/(\rho\Delta_T)$ because $\bar{f}(-1) = v/2$. □

Corollary A.3 (Counts under purchases only)**.** *Take an equilibrium in which class L abstains and class H buys with probability $\sigma_H(1) > 0$, and write $s_+ = \lambda\sigma_H(1)$.*

(a) Then $\bar{f}(-1) = b$, $\bar{f}(1) = b + \pi s_+$, $\mathbb{E}[X] = \pi s_+$ and $v = 2b + \pi s_+$. At every prior, the counts identify $b = (v - \mathbb{E}[X])/2$ and the share $\mathbb{E}[X]/v$ of nonzero orders that the informed trader places. At a known prior they identify $s_+ = \mathbb{E}[X]/\pi$; if class H buys with probability one, $\lambda = \mathbb{E}[X]/\pi$.

(b) Suppose that an indifferent informed trader may mix over its best orders and that class H mixes, $0 < \sigma_H(1) < 1$. For every $\lambda' \in [s_+, 1 - 3b]$, the slot (λ', β') with $\beta' = b/(1-\lambda')$, with

$\sigma'_H(1) = s_+/\lambda'$ and with class L abstaining, has an equilibrium with the same class-conditional order distributions, and so the same posteriors, prices, entry and acquisition probabilities. The counts therefore identify b and, at a known prior, s_+; even with the prices, they do not identify λ. On a mixed purchase, the indifference of class H gives $s_+ = (b/\pi)\big[e(1)(1-\pi)\Delta_T/k - 1\big]$, which is $(b/\pi)\big[(1-\pi)\Delta_T/k - 1\big]$ when every rival prepares after a purchase.

Proof of Corollary A.3. By Lemma A.3(a) no class places a wrong-signed order, so class H places the order 0 with probability $1 - \sigma_H(1)$. The class-conditional order distributions on $(-1, 0, 1)$ are

$$f_H = (b,\ 1 - 2b - s_+,\ b + s_+), \qquad f_L = (b,\ 1 - 2b,\ b).$$

(a) Averaging with the weights π and $1 - \pi$ gives $\bar{f}(-1) = b$ and $\bar{f}(1) = b + \pi s_+$, so $\mathbb{E}[X] = \pi s_+$ and $v = 2b + \pi s_+$. Hence $b = (v - \mathbb{E}[X])/2$. The informed trader places a nonzero order with probability $\pi s_+ = \mathbb{E}[X]$, so its share of nonzero orders is $\mathbb{E}[X]/v$. At a known prior, $s_+ = \mathbb{E}[X]/\pi$, and $s_+ = \lambda$ when $\sigma_H(1) = 1$.

(b) The original slot has $\beta \le 1/3$, that is $\lambda \le 1 - 3b$, and $s_+ = \lambda\sigma_H(1) < \lambda$, so the segment $[s_+, 1 - 3b]$ is nondegenerate and contains λ. Take λ' in it. Then $\beta' = b/(1 - \lambda')$ lies in $(0, 1/3]$ because $1 - \lambda' \ge 3b$, the liquidity mass is $(1 - \lambda')\beta' = b$, and $\sigma'_H(1) = s_+/\lambda'$ lies in $(0, 1]$ because $\lambda' \ge s_+$. The class-conditional order distributions of this slot are f_H and f_L above, since they depend only on b and s_+. By (A.3) the posteriors are the same, so keep the original entry $e(x)$ after each order: it meets Lemma A.2(d) at these posteriors under either tie rule for the rival. Prices $t_0 + e(x)\, w_F(\mu(x))$, the preparation rule and the price beliefs are then the same as well. The payoffs of Lemma A.3 depend only on posteriors and entry, so they do not change, and by Lemma A.3(d) the factor λ' changes no comparison. In the original equilibrium class H mixes, so the purchase and abstention are its two best orders, and an indifferent informed trader who may mix over them can buy with probability $\sigma'_H(1)$; abstention stays optimal for class L. So the slot (λ', β') has an equilibrium with the same order distributions, posteriors, prices and entry, and by Proposition A.12(i) the same acquisition probability $e(x)\, W_F(\mu(x))$ after each order. The counts and the prices are therefore the same at every λ' in the segment; part (a) gives b and, at a known prior, s_+.

On a mixed purchase, class H is indifferent, $K_H = e(1)\Delta_T\,[1 - \mu(1)] = k$, with $1 - \mu(1) = (1 - \pi)b/(b + \pi s_+)$ by Bayes' rule. Solving $e(1)\Delta_T(1 - \pi)b = k\,(b + \pi s_+)$ for s_+ gives the formula, and $e(1) = 1$ gives its second form. □

Let $\mathbf{1}_{\text{prep}}$ and $\mathbf{1}_{\text{acq}}$ indicate that the rival prepares and that it acquires the target. For a class j, $W_j(F) = \Pr(R < \theta)$ with θ drawn from G_j is the probability that a prepared class-j rival outbids the incumbent. Write $\Omega(F) = W_H(F) - W_L(F)$ and $W_F(\mu) = W_L(F) + \mu\,\Omega(F)$.

Proposition A.12 (Orders predict the winner)**.**

(i) In every equilibrium, the probability that the rival acquires the target after the order x, given preparation, is $W_F(\mu(x))$, and $\mathbb{E}[\mathbf{1}_{\text{acq}} \mid X = x] = e(x)\, W_F(\mu(x))$. Also $\Omega(F) = \mathbb{E}_F[D(R)] \geq 0$, and $\Omega(F) > 0$ exactly when $\Pr_F\{D(R) > 0\} > 0$.

(ii) Under full two-sided trade, $W_F(M) - W_F(m) = (M - m)\,\Omega(F) \geq 0$, positive exactly when $\Pr_F\{D(R) > 0\} > 0$. The prices rise strictly with the order, so the win probability of a prepared rival rises with the price, strictly when $\Pr_F\{D(R) > 0\} > 0$.

(iii) Under full two-sided trade with $\pi < \tau$,

$$\begin{aligned}\operatorname{Cov}(X, \mathbf{1}_{\text{prep}}) &= [e(1) - \rho]\, \bar{f}(1)\, (1 - \mathbb{E}[X]),\\ \operatorname{Cov}(X, \mathbf{1}_{\text{acq}}) &= 2\pi(1 - \pi)\rho\lambda\, \Omega(F) + [e(1) - \rho]\, \bar{f}(1)\, (1 - \mathbb{E}[X])\, W_F(M).\end{aligned}$$

If $B_F(M) < c_H$, entry is ρ after every order, $\operatorname{Cov}(X, \mathbf{1}_{\text{prep}}) = 0$, and $\operatorname{Cov}(X, \mathbf{1}_{\text{acq}}) = 2\pi(1 - \pi)\rho\lambda\, \Omega(F)$, which is positive exactly when $\Pr_F\{D(R) > 0\} > 0$.

(iv) Under full two-sided trade with $\pi < \tau$,

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\text{acq}} \mid X = 1] - 2\,\mathbb{E}[\mathbf{1}_{\text{acq}} \mid X = 0] + \mathbb{E}[\mathbf{1}_{\text{acq}} \mid X = -1]\\ &\qquad = \rho\,(M + m - 2\pi)\,\Omega(F) + [e(1) - \rho]\, W_F(M).\end{aligned}$$

At the prior $1/2$, $M + m = 1 = 2\pi$. If also an indifferent high-cost rival prepares, then $\operatorname{Cov}(X, \mathbf{1}_{\text{acq}}) = \rho\lambda\, \Omega(F)/2 + (1 - \rho)\, \chi_F\, \bar{f}(1)\, W_F(M)$, and the second difference equals $(1 - \rho)\, \chi_F\, W_F(M)$, which is positive exactly when a purchase recruits.

(v) Let F_r be a path, hold the prior and the slot fixed, and let class H always outbid the incumbent: $W_H(F_r) = 1$ at every strength. At strengths with full two-sided trade, the winner gap $W_{F_r}(M) - W_{F_r}(m) = (M - m)\,[1 - W_L(F_r)]$ is nondecreasing in r, and it is larger at r' than at $r < r'$ exactly when $W_L(F_{r'}) < W_L(F_r)$.

(vi) If F_r is uniform on $[0, r]$ with $r \geq p$, then $\Omega(F_r) = \Gamma(r)/r$, and its right derivative in r is $[D(r) - \Omega(F_r)]/r$. So $\Omega(F_r)$, and with it the full-trade winner gap $(M - m)\,\Omega(F_r)$ at a fixed prior and slot, is nondecreasing on an interval of strengths exactly when $D(r) \geq \Omega(F_r)$ at almost every strength of the interval, and it rises strictly on an interval where $D(r) > \Omega(F_r)$ at almost every strength.

Proof of Proposition A.12. (i) By Step 0 of the proof of Lemma A.2, given the class j and the order x the rival prepares with probability $e(x)$, independently of (θ, R). Preparation therefore

leaves the class posterior at $\mu(x)$, and given the class it leaves (θ, R) at its prior distribution. Bids are truthful by (E6), and every value meets the reserve, so a prepared rival acquires the target exactly when $R < \theta$; ties have probability zero because F is continuous. Given the class j and preparation, it acquires with probability $W_j(F)$, and given preparation alone with probability $(1-\mu(x))W_L(F) + \mu(x)W_H(F) = W_F(\mu(x))$. Without preparation it does not acquire, so $\mathbb{E}[\mathbf{1}_{\text{acq}} \mid X = x] = e(x)\, W_F(\mu(x))$. Also $W_j(F) = \int \Pr_j(\theta > y)\, dF(y) = \int [1-G_j(y)]\, dF(y)$, since $\Pr_j(\theta > y)$ and $1 - G_j(y)$ differ at countably many y, which F does not charge. Hence $\Omega(F) = \int [G_L(y) - G_H(y)]\, dF(y) = \mathbb{E}_F[D(R)]$. The class ranking gives $D \geq 0$, so $\Omega(F) \geq 0$, with $\Omega(F) > 0$ exactly when $D(R) > 0$ with positive probability.

(ii) Under full two-sided trade the posteriors after a purchase and a sale are M and m, and W_F is affine with slope $\Omega(F)$, so $W_F(M) - W_F(m) = (M-m)\,\Omega(F)$, with $M > m$. By Lemma A.2(e) the prices rise strictly with the posterior, so $P(-1) < P(0) < P(1)$, and the win probability $W_F(\mu(x))$ of a prepared rival is a nondecreasing function of the price, strictly increasing when $\Omega(F) > 0$.

(iii) For any indicator Y, $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, \mathbb{E}[Y \mid X])$. With $\pi < \tau$, entry after the orders $(-1, 0, 1)$ is $(\rho, \rho, e(1))$, so $\mathbb{E}[\mathbf{1}_{\text{prep}} \mid X] = \rho + [e(1) - \rho]\, \mathbf{1}\{X = 1\}$. For any number c, $\mathrm{Cov}(X, c\, \mathbf{1}\{X = 1\}) = c\, [\bar{f}(1) - \mathbb{E}[X]\, \bar{f}(1)] = c\, \bar{f}(1)\, (1 - \mathbb{E}[X])$, which gives the first line. By (i), $\mathbb{E}[\mathbf{1}_{\text{acq}} \mid X] = \rho\, W_F(\mu(X)) + [e(1) - \rho]\, W_F(M)\, \mathbf{1}\{X = 1\}$, and $\mathrm{Cov}(X, \rho\, W_F(\mu(X))) = \rho\, \Omega(F)\, \mathrm{Cov}(X, \mu(X))$. Since $\mu(X) = \Pr(H \mid X)$, $\mathbb{E}[X\mu(X)] = \mathbb{E}[X\, \mathbf{1}\{j = H\}] = \pi\, \mathbb{E}[X \mid H] = \pi\lambda$ under full two-sided trade, so $\mathrm{Cov}(X, \mu(X)) = \pi\lambda - \lambda(2\pi - 1)\pi = 2\pi(1-\pi)\lambda$. Adding the two terms gives the second line. If $B_F(M) < c_H$, then $M < \tau$, every posterior lies below τ, entry is ρ after every order, and $e(1) = \rho$ in both lines; positivity follows from (i).

(iv) By (i), with entry $(\rho, \rho, e(1))$ and posteriors (m, π, M), the second difference is $e(1)W_F(M) - 2\rho W_F(\pi) + \rho W_F(m) = \rho\, [W_F(M) - 2W_F(\pi) + W_F(m)] + [e(1) - \rho]\, W_F(M)$, and $W_F(M) - 2W_F(\pi) + W_F(m) = (M + m - 2\pi)\, \Omega(F)$ because W_F is affine. At the prior $1/2$, $M + m = 1$ by the standing paragraph and $\mathbb{E}[X] = 0$. When an indifferent high-cost rival prepares, $e(1) = \rho + (1-\rho)\chi_F$, because $B_F(M) \geq c_H$ exactly when $M \geq \tau$. Substituting in (iii) and in the second difference gives the two expressions. If $\chi_F = 1$, then $(1 - M)g_L + Mg_H = B_F(M) \geq c_H > 0$, so some class has positive profit $\mathbb{E}_j[(\theta - \max\{p, R\})_+]$, which is positive only when $\Pr(R < \theta) > 0$; that class has $W_j(F) > 0$, and $W_F(M)$, which puts positive weight on both classes, is positive. So the second difference is positive exactly when $\chi_F = 1$, that is, exactly when a purchase recruits.

(v) With $W_H(F_r) = 1$, $\Omega(F_r) = 1 - W_L(F_r)$, and (ii) gives the winner gap. Since $W_L(F_r) = \mathbb{E}_L[F_r(\theta)]$ and $F_{r'} \leq F_r$ for $r' > r$, W_L is nonincreasing along the path. The prior and the

slot are fixed, so $M - m > 0$ is fixed, and the gap at r' exceeds the gap at r exactly when $W_L(F_{r'}) < W_L(F_r)$.

(vi) Write $\Gamma(x) = \int_p^x D(y)\,dy$ for $x \geq p$, the class difference in expected proceeds at the incumbent value x (Appendix A.1). Under the class ranking $G_H = G_L = 0$ below p, so $D = 0$ there, and for the uniform distribution on $[0, r]$ with $r \geq p$, $\Omega(F_r) = \mathbb{E}_{F_r}[D(R)] = \frac{1}{r}\int_0^r D(y)\,dy = \Gamma(r)/r$. Since D is right-continuous, Γ has the right derivative $D(r)$, and the quotient rule gives the right derivative $[rD(r) - \Gamma(r)]/r^2 = [D(r) - \Omega(F_r)]/r$. Since $0 \leq D \leq 1$, Γ is Lipschitz, so $\Omega(F_r)$ is absolutely continuous on every compact interval of positive strengths, and its derivative equals $[D(r) - \Omega(F_r)]/r$ at every continuity point of D, hence almost everywhere, because D is a difference of two monotone functions. An absolutely continuous function is nondecreasing on an interval exactly when its derivative is nonnegative almost everywhere there, and it rises strictly where its derivative is positive almost everywhere. The prior and the slot are fixed, so the full-trade winner gap $(M - m)\,\Omega(F_r)$ inherits both properties. □

Add a public event after the price that reveals only whether the rival prepared, and nothing else. Market makers then price again.

Proposition A.13 (Announcement returns)**.** *Take any equilibrium, at any prior, under either tie rule for the rival.*

(i) After the order x, the price changes by $\Delta_x^1 = [1 - e(x)]\,w_F(\mu(x))$ when the event reports preparation, and by $\Delta_x^0 = -e(x)\,w_F(\mu(x))$ when it reports no preparation. The second report has positive probability exactly when $e(x) < 1$.

(ii) After an order x with $e(x) < 1$, the mean change over both reports, with their probabilities, is zero.

(iii) Pool all orders, and take the sample of preparation reports. Its mean price change is

$$\sum_x \Pr(X = x \mid \textit{preparation})\,\Delta_x^1 = \frac{\sum_x \bar{f}(x)\,e(x)\,[1 - e(x)]\,w_F(\mu(x))}{\sum_x \bar{f}(x)\,e(x)},$$

which is positive exactly when entry is below one after some order, that is, exactly when preparation is uncertain. If $B_F(\pi) < c_H$, entry is ρ after some order, and the mean is positive.

(iv) The expected price before the order opportunity minus the price after no order is the weighted impact difference, $\mathbb{E}[P(X)] - P(0) = \bar{f}(1)I_+ - \bar{f}(-1)I_-$, which parts (i) to (iii) of Proposition A.10 evaluate. Under full two-sided trade the posterior after no order is the prior.

Proof of Proposition A.13. After the order x, the event reveals whether the rival prepared, and the market makers price at the expected value of the target given the order and the report.

(i) By the proof of Proposition A.12(i), preparation leaves the class posterior at $\mu(x)$ and, given the class j, leaves (θ, R) at its prior distribution. With truthful bids (E6), expected proceeds in class j after preparation are t_j, so the price after a report of preparation is $(1-\mu(x))t_L + \mu(x)t_H = t_0 + w_F(\mu(x))$. Without preparation the target receives the proceeds without entry, with mean t_0 in both classes. Subtracting $P(x) = t_0 + e(x)\, w_F(\mu(x))$ gives Δ_x^1 and Δ_x^0. Given the order x, the report of no preparation has probability $1 - e(x)$, which is positive exactly when $e(x) < 1$.

(ii) When $e(x) < 1$, $e(x)\Delta_x^1 + [1-e(x)]\Delta_x^0 = e(x)[1-e(x)]\, w_F(\mu(x)) - [1-e(x)]\, e(x)\, w_F(\mu(x)) = 0$.

(iii) The probability of the order x given preparation is $\bar{f}(x)e(x)/\sum_{x'} \bar{f}(x')e(x')$, which gives the display. Every order has $\bar{f}(x) > 0$ by Lemma A.1(a), entry $e(x) \geq \rho > 0$, and an interior posterior, so $w_F(\mu(x)) > 0$. Every term of the numerator is therefore nonnegative, and the numerator is positive exactly when $e(x) < 1$ after some order. Every order has positive probability, so this is exactly when preparation is uncertain. If $B_F(\pi) < c_H$, Bayes' rule and the affine form of B_F give $\sum_x \bar{f}(x)\, B_F(\mu(x)) = B_F(\pi) < c_H$, so some order has $B_F(\mu(x)) < c_H$, that is $\mu(x) < \tau$, and Lemma A.2(d) gives $e(x) = \rho < 1$ after it, under both tie rules for the rival.

(iv) The identity is part (i) of Proposition A.10, which holds in every equilibrium. Under full two-sided trade, Lemma A.1(d) gives $\mu(0) = \pi$. □

The example. *Take the example at the prior* $1/2$*: the incumbent is uniform on* $[0, r]$*, the classes are point masses at* h *and* ℓ*, and the slot lies in the band* $\tau(r_1) < M < \tau(r_2)$ *of Corollary 1. Unless a part names another tie rule, an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares. The numbers are those of the example's slot.*

(a) Full two-sided trade. *For* $1 < r < h$, $B_r(1/2) = 5 - r/4 + 1/(8r) < c_H$, *so* $\tau(r) > 1/2$. *The strength* r_S *solves* $\rho\Delta_T(r)\, m = k$: $r_S = \mathfrak{r}\big(k/(\rho m)\big)$ *with* $k/(\rho m) = 0.297463$, *and* $r_S \approx 2.124148$. *For* $r_S < r < h$, *every equilibrium has full two-sided trade, under both tie rules for the informed trader and both tie rules for the rival, whether or not the rival sees the price. Its posteriors are* $(m, 1/2, M)$*; its order frequencies are* $\bar{f}(\pm 1) = b + \lambda/2 = 0.363676$ *and* $\bar{f}(0) = 1 - 2b - \lambda$*; its volume is* $v = 0.727352$*;* $\mathbb{E}[X] = 0$*; and the informed share is* $\vartheta = M - m = 0.462117$. *By Lemma A.11,* $r_S < r_1 < r_C < r_2 < h$, *and* $r_C - r_S = 1.468510$.

(b) Recruitment and entry. *The top profit* $B_r(M)$ *falls strictly in* r *for* $r > \ell$, *equals* $9M + 3/8$ *at* $r = \ell$, *and lies below* c_H *at* $r = h$. *In the band,* $M > \tau(r_1) = 141/200 > 5/8$, *so the entry ceiling* r_C *is the unique strength with* $B_r(M) = c_H$, *and* $\chi_r = 1$ *exactly when* $r \leq r_C$. *When an*

indifferent high-cost rival prepares, entry after the orders $(-1, 0, 1)$ *is* $(\rho, \rho, \rho + (1-\rho)\chi_r)$ *on* (r_S, h)*, and entry is* $\rho + (1-\rho)\chi_r\,(b + \lambda/2)$*. From* r_0*, where the orders are zero and entry is* ρ*, to* r_1*, entry rises by* $(1-\rho)(b+\lambda/2) = 0.272757$*. With a hidden price, entry is* ρ *after every order.*

(c) The price gap. *Here* $w_r(\mu) = 1/2 - 1/(8r) + \Delta_T(r)\,\mu$ *with* $\Delta_T(r) = (r-1)^2/(2r)$*, and both* $1/2 - 1/(8r)$ *and* $\Delta_T(r)$ *rise strictly in* r *for* $r > \ell$*; also, for* $r \geq \ell$*,* $\min\{\ell, r\}^2 - 3p^2 = \ell^2 - 3p^2 = 1/4 > 0$*, the condition of Proposition A.10(vi)(b). At the prior* $1/2$*,* $\bar{f}(1) = \bar{f}(-1)$*, and the weighted test reads* $I_+ - I_- = (1-\rho)\chi_r\, w_r(M)$*, so* $I_+ > I_-$ *exactly when* $r \leq r_C$*. On* $(r_S, r_C]$*, both components of the price gap, both impacts, the price gap and the weighted impact difference rise strictly with* r*. On* (r_C, h)*, the entry component and the weighted impact difference are zero, and the class component, both impacts and the price gap rise strictly. At* r_1 *the class component is* 0.077020 *and the entry component* 0.709279*, so the price gap is* 0.786299*, of which the entry component is* 90.2%*; the weighted impact difference* $\mathbb{E}[P(X)] - P(0)$ *is* 0.257948*. With a hidden price, the price gap at* r_1 *is the class component* 0.077020*. At* r_C *the price gap falls from* 0.969913 *to* 0.108078*, by* 0.861835*, and at* r_2 *it is the class component* 0.108469*. When an indifferent informed trader may mix over its best orders and an indifferent high-cost rival may prepare with any probability, entry after a purchase at* r_C *can take any value in* $[\rho, 1]$*, and the price gap at* r_C *any value from* 0.108078 *to* 0.969913*.*

(d) Identification. *The counts identify only the volume. When an indifferent informed trader may mix over its best orders, the informed shares that the counts and full two-sided trade at* r_1 *allow form the interval from* $3 - 2/v \approx 0.250299$ *to* $1 - 2k/(\rho\Delta_T(r_1)) = 0.760000$*, both ends included; when an indifferent informed trader abstains, the right end is excluded. At* r_2 *the right end is* 0.829586*. The shares with strict recruitment at* r_1 *and strict exclusion at* r_2 *form the band* $(2\tau(r_1) - 1,\ 2\tau(r_2) - 1) = (0.410000, 0.462784)$*. Each share in the band belongs to a slot with the example's counts and the same entry at* r_1 *and at* r_2*, and full two-sided trade is the unique outcome of that slot at both strengths. The slot with share* 0.444444 *is one of them; its price gap at* r_1 *is* 0.778935*, against* 0.786299*. On* (r_S, h) *the sale impact identifies the share,* $\vartheta = 2I_-/(\rho\Delta_T(r))$*.*

(e) Purchases only. *A full purchase with class L abstaining is an equilibrium exactly on* $[r_F, r_N]$*, and a mixed purchase that recruits is an equilibrium on* (r_A, r_F)*, each beside an uninformative market (Corollary 2). On* $[r_F, r_N]$ *the counts identify* $\lambda = 2\mathbb{E}[X]$*. On* (r_A, r_F)*,* $s_+ = b\,[\Delta_T(r)/k - 2]$*, and the counts and the prices do not identify* λ*.*

(f) Winners. *For* $\ell \leq r \leq h$*,* $W_H = 1$*,* $W_L = \ell/r$ *and* $\Omega = 1 - \ell/r$*, so a prepared rival wins after the order* x *with probability* $\mu(x) + [1 - \mu(x)]\,\ell/r$*. Under full two-sided trade the winner gap*

$\vartheta(1-\ell/r)$ rises strictly with r: class H always outbids the incumbent and W_L falls strictly, and $D(r)-\Omega=\ell/r>0$ for $\ell\le r<h$, which is $1/3$ at r_1 and $5/18$ at r_2. The winner gap is 0.308078 at r_1 and 0.333751 at r_2. The covariance of the order with entry is 0.272757 at r_1 and zero at r_2. Its covariance with a win by the rival is 0.251863 at r_1, the class term $\rho\lambda\Omega/2=0.028010$ plus the recruitment term 0.223853, and 0.030344 at r_2, the class term alone. The second difference of the win probability across the orders is 0.615529 at r_1 and zero at r_2.

(g) Announcements. *Since $B_r(1/2)$ falls in r, $c_H-B_r(1/2)\ge c_H-B_{r_0}(1/2)=1.195833$ for every $r\ge r_0$, so a sample of preparation reports has a positive mean price change in every equilibrium at every such strength. At r_1 the mean is 0.160592; the expected price before the order opportunity is 0.872531, and the price after no order is 0.614583, lower by 0.257948. At r_2 the mean is 0.701042, and the price after no order equals the expected price.*

(h) Two clusters. *Let $r_F<r<r_N$, so $r<r_C$. When an indifferent informed trader may mix over its best orders and an indifferent high-cost rival prepares, Corollary 2 gives exactly two equilibria: an uninformative market, and a full purchase with class L abstaining. In the second, $\bar f(1)=b+\lambda/2$ and $\bar f(-1)=b$, so $\mathbb{E}[X]=\lambda/2=0.168061$. The posterior after a sale is $1/2$, and after no order it is $\mu(0)=(1-2b-\lambda)/(2-4b-\lambda)$, with $1/2-\mu(0)=\lambda/[4(1-v)]>0$ for the volume $v=2b+\lambda/2$ of this equilibrium. Both orders have entry ρ, so $P(-1)-P(0)=\rho\Delta_T\lambda/[4(1-v)]>0$, which is 0.005363 at $r=1.60$; after a purchase every rival prepares. In the uninformative market the three prices coincide and entry after a purchase is ρ. When an indifferent high-cost rival may prepare with any probability, the threshold pair adds a third equilibrium: at $r=1.60$, class H buys with probability 0.492845, entry after a purchase is 0.506106, the mean order is 0.082828, the low-class sale earns -0.005938, and entry is 0.321311.*

Proof of the statements on the example. At the prior $1/2$, $1-M=m$, so $\kappa(1/2)=m$, and $\bar f(\pm1)=b+\lambda/2$ under full two-sided trade. For $p<\ell<r<h$, Appendix A.1 gives

$$t_0=p\Big(1-\frac{p}{r}\Big),\quad t_L=\ell-\frac{\ell^2-p^2}{2r},\quad g_H=h-\frac{r}{2}-\frac{p^2}{2r},\quad g_L=\frac{\ell^2-p^2}{2r},\quad \Delta_T(r)=\frac{(r-\ell)^2}{2r},$$

and Δ_T is zero at ℓ and strictly increasing above ℓ.

(a) Gross profit at the prior is $B_r(1/2)=(g_H+g_L)/2=h/2-r/4+(\ell^2-2p^2)/(4r)$, which is $5-r/4+1/(8r)$ at the example's values. It falls strictly in r, and at $r=\ell$ it equals $h/2-p^2/(2\ell)<h/2<c_H$, so $B_r(1/2)<c_H$ for $\ell<r<h$. There $\gamma=g_H-g_L=h-r/2-\ell^2/(2r)>h-r>0$, so $\tau(r)>1/2$. The strength r_S is the unique strength above ℓ

at which $\rho\Delta_T(r)\, m = k$, and for $r > r_S$, $\rho\kappa(1/2)\Delta_T(r) = \rho m\Delta_T(r) > k$. Proposition A.10(iv) gives full two-sided trade in every equilibrium for $r_S < r < h$, under both tie rules for the informed trader, whether or not the rival sees the price; the rival's rule enters no step. The standing paragraph at the prior $1/2$ gives the posteriors, the frequencies, the volume, the mean order and the share. Lemma A.11 gives $r_S < r_1 < r_C < r_2$, and $r_2 < h$.

(b) By Appendix A.1, $\partial B_r(\mu)/\partial r < 0$ for every μ when $\ell < r < h$, so $B_r(M)$ falls strictly there. At $r = \ell$, $g_L = (\ell^2 - p^2)/(2\ell)$ and $g_H = h - \ell/2 - p^2/(2\ell)$, which give $9M + 3/8$ at the example's values; it exceeds c_H exactly when $M > 5/8$. At $r = h$, $B_h(M) < g_H = h/2 - p^2/(2h) < h/2 < c_H$, because $g_L < g_H$ and $M < 1$. In the band, $M > \tau(r_1)$, and $\tau(r_1) = 141/200 > 5/8$. So $B_r(M)$ crosses c_H once on (ℓ, h), at r_C. Since $B_r(M)$ is nonincreasing in r at every strength (Proposition 1(a)), $\chi_r = 1$ exactly when $r \le r_C$. On (r_S, h), the posteriors after a sale and after no order are $m < 1/2 < \tau(r)$, so entry after them is ρ, and when an indifferent high-cost rival prepares, entry after a purchase is $\rho + (1-\rho)\chi_r$, because $B_r(M) \ge c_H$ exactly when $M \ge \tau(r)$. Weighting by the frequencies gives $\rho + (1-\rho)\chi_r\,(b + \lambda/2)$. At r_0, Corollary 1 gives an uninformative market with entry ρ; at r_1, $\chi_{r_1} = 1$. With a hidden price, entry is ρ after every order by Proposition A.10(iv), because $1/2 < \tau(r)$.

(c) The entry gain against r is $w_r(\mu) = t_L - t_0 + \Delta_T(r)\mu$, with $t_L - t_0 = \ell - p - (\ell^2 - 3p^2)/(2r)$, which is $1/2 - 1/(8r)$ at the example's values. Since $\ell^2 - 3p^2 > 0$ and Δ_T rises strictly above ℓ, both terms rise strictly. On a path of strengths in (ℓ, h), the lowest value of the rival is ℓ and the lowest strength lies above ℓ, so $\ell^2 - 3p^2 > 0$ is the condition $p < \min\{\underline{\theta}, \underline{r}\}/\sqrt{3}$ of Proposition A.10(vi)(b). At the prior $1/2$, $\bar{f}(1) = \bar{f}(-1)$, so Proposition A.10(ii), with $1/2 < \tau(r)$, gives $I_+ - I_- = [e(1) - \rho]\, w_r(M) = (1-\rho)\chi_r\, w_r(M)$ by (b), and $I_+ > I_-$ exactly when $\chi_r = 1$. On (r_S, h) full two-sided trade holds, $1/2 < \tau(r)$, Δ_T rises strictly and $w_r(M)$ rises strictly, so Proposition A.10(vi)(c) applies on $(r_S, r_C]$, where $\chi_r = 1$, and on (r_C, h), where $\chi_r = 0$. By Proposition A.10(iv) and (ii), at r_1 the class component is $\rho\Delta_T(r_1)\vartheta$, the entry component is $(1-\rho)\, w_{r_1}(M)$, and the weighted impact difference is $(1-\rho)(b + \lambda/2)\, w_{r_1}(M)$; with a hidden price the price gap is the class component. The uniform distributions have $F_r(y) = \min\{y/r, 1\}$, continuous in r, and (r_S, h) is a neighborhood of r_C on which full two-sided trade and $1/2 < \tau(r)$ hold, so Proposition A.10(vi)(d) gives the fall $(1-\rho)\, w_{r_C}(M)$, from $\rho\Delta_T(r_C)\vartheta + (1-\rho)\, w_{r_C}(M)$ to $\rho\Delta_T(r_C)\vartheta$. At $r_2 > r_C$, $\chi_{r_2} = 0$. When an indifferent high-cost rival may prepare with any probability, the last part of Proposition A.10(vi)(d) applies, with an indifferent informed trader who may mix. Substituting the example's slot in these expressions gives the values stated in (c).

(d) Since $\tau(r) > 1/2$ at r_1 and r_2, Proposition A.11(b) applies at both. It gives the left end $3 - 2/v$ from the counts, and the right ends $1 - 2k/(\rho\Delta_T(r_1))$ and $1 - 2k/(\rho\Delta_T(r_2))$ from

full two-sided trade at each strength, both included when an indifferent informed trader may mix and the right end excluded when an indifferent informed trader abstains. At the prior $1/2$, a slot with share ϑ' has $M' = (1+\vartheta')/2$. Strict recruitment at r_1 is $M' > \tau(r_1)$, and strict exclusion at r_2 is $M' < \tau(r_2)$; together they are the band. Its left end exceeds $3 - 2/v$: the inequality $2\tau(r_1) - 1 > 3 - 2/v$ is $v < 2/[4 - 2\tau(r_1)] = 200/259$, and the example's volume lies below this bound by 0.044849. So Proposition A.11(b) gives each share in the band a slot with the example's counts, and its entry formula gives entry $\rho + (1-\rho)v/2$ at r_1 and ρ at r_2, as at the example's slot, whose top posterior lies in the band too. Such a slot has $m' = (1-\vartheta')/2 > 1 - \tau(r_2)$, so at r_1 and at r_2

$$\rho\, m'\, \Delta_T(r) - k > \rho\, [1 - \tau(r_2)]\, \Delta_T(r_1) - k = 0.024768 > 0,$$

because $\Delta_T(r_2) > \Delta_T(r_1)$. Proposition A.10(iv) then gives full two-sided trade as the unique outcome of the slot at both strengths. The share 0.444444 lies in the band, and by Proposition A.10(iv) its price gap at r_1 is $\rho\Delta_T(r_1)\vartheta' + (1-\rho)\, w_{r_1}(M')$. On (r_S, h), entry after a sale and after no order is ρ, and Proposition A.11(c) at the prior $1/2$ gives the share from the sale impact.

(e) Corollary 2 and Table A.2 give the two equilibria and their intervals. At a full purchase with class L abstaining, both classes sell with probability b, so a sale meets the posterior $1/2$ and earns $\rho\Delta_T(r)/2 - k$, which is nonpositive exactly when $r \le r_N$; for $r < r_C$ the purchase recruits and earns $\Delta_T(r)(1-M) - k$, which is nonnegative exactly when $r \ge r_F$. On $[r_F, r_N]$ class H buys with probability one, and Corollary A.3(a) at the prior $1/2$ gives $\lambda = 2\mathbb{E}[X]$. On (r_A, r_F) class H mixes, an indifferent informed trader may mix, and the posterior after a purchase exceeds $\tau(r)$, so every rival prepares after a purchase. Corollary A.3(b) gives $s_+ = 2b\,[\Delta_T(r)/(2k) - 1] = b\,[\Delta_T(r)/k - 2]$ and the failure to identify λ.

(f) For $\ell \le r \le h$, $R \le r \le h$, so a prepared high-class rival wins with probability one, a prepared low-class one wins exactly when $R < \ell$, and $W_L = \ell/r$. Proposition A.12(i) gives the win probability after each order, and (ii) gives the winner gap $(M-m)\,\Omega = \vartheta(1 - \ell/r)$ under full two-sided trade. Since $W_H = 1$ and $W_L = \ell/r$ falls strictly, Proposition A.12(v) makes the gap rise strictly. Here $D(y) = \mathbf{1}\{\ell \le y < h\}$, so $D(r) - \Omega = \ell/r > 0$ for $\ell \le r < h$, the condition of Proposition A.12(vi). On (r_S, h) full two-sided trade holds with $1/2 < \tau(r)$ and $\mathbb{E}[X] = 0$, so Proposition A.12(iii) and (iv) give $\mathrm{Cov}(X, \mathbf{1}_{\mathrm{prep}}) = (1-\rho)\chi_r(b + \lambda/2)$, $\mathrm{Cov}(X, \mathbf{1}_{\mathrm{acq}}) = \rho\lambda\,\Omega/2 + (1-\rho)\chi_r\,(b + \lambda/2)\, W_{F_r}(M)$, and the second difference $(1-\rho)\chi_r\, W_{F_r}(M)$, with $\chi_{r_1} = 1$ and $\chi_{r_2} = 0$. Substituting the example's slot gives the values stated in (f).

(g) The incumbent distributions strengthen with r, so $B_r(1/2)$ falls in r by Proposition 1(a), and

$B_r(1/2) < c_H$ for $r \geq r_0$. Proposition A.13(iii) gives a positive mean in every equilibrium at each such strength. At r_1 entry after the orders $(-1, 0, 1)$ is $(\rho, \rho, 1)$, and at r_2 it is ρ after every order; the display of Proposition A.13(iii) gives the two means. By Proposition A.13(iv) and Proposition A.10(ii), the expected price minus the price after no order is $(1-\rho)\chi_r\,(b + \lambda/2)\,w_r(M)$, the weighted impact difference of (c), which is zero at r_2. Substituting the example's slot in $P(x) = t_0 + e(x)\,w_r(\mu(x))$ gives the prices stated in (g).

(h) Lemma A.11 gives $r_N < r_C$, so $r < r_C$. Under the default tie rules, Corollary 2 and Table A.2 give exactly two equilibria on (r_F, r_N): an uninformative market and a full purchase with class L abstaining. In the second, Corollary A.3(a) at the prior $1/2$ with $s_+ = \lambda$ gives $\bar{f}(1)$, $\bar{f}(-1)$ and $\mathbb{E}[X] = \lambda/2$. Both classes sell with probability b, so the posterior after a sale is $1/2$. After no order, the class probabilities are $1 - 2b - \lambda$ and $1 - 2b$, which give $\mu(0)$ and $1/2 - \mu(0) = \lambda/[2(2 - 4b - \lambda)] = \lambda/[4(1 - v)]$. Both posteriors lie below $\tau(r)$, so entry after both orders is ρ, and $P(-1) - P(0) = \rho\,[w_r(1/2) - w_r(\mu(0))] = \rho\Delta_T(r)\,[1/2 - \mu(0)]$. After a purchase the posterior is $M > \tau(r)$, so every rival prepares. In the uninformative market every posterior is $1/2$, entry is ρ after every order, and the three prices coincide. When an indifferent high-cost rival may prepare with any probability, the threshold pair of Appendix A.4 is an equilibrium on $(r_A, r_J]$, which contains $[r_F, r_N]$ by Lemma A.11. In it class L abstains; class H buys at the rate $s_+ = b\,[2\tau(r) - 1]/[1 - \tau(r)]$, at which the posterior after a purchase equals $\tau(r)$; and entry after a purchase is $e(1) = k/J(r)$, with $J(r) = \Delta_T(r)\,[1 - \tau(r)]$, at which class H is indifferent. So class H buys with probability s_+/λ, the mean order is $s_+/2$ by Corollary A.3(a), the low-class sale earns $\rho\Delta_T(r)/2 - k < 0$, because the posterior after a sale is $1/2$ and $r < r_N$, and entry is $\rho + [e(1) - \rho]\,(b + s_+/2)$. Substituting the example's slot at $r = 1.60$ gives the values stated in (h). □

A.9 Parameters of the example

This subsection declares the inputs of every computation and records the margin of every strict hypothesis in the example. Table A.5 gives the example's parameters, following Section 2: a declared value is a primitive, and a derived value follows from its definition. The derived values keep their unrounded definitions in every computation; the decimals are displays.

The signal example. The signal example of Section 6.3 keeps the slot (u, β, λ), the values h and ℓ, the reserve p and the prior π of the example, and it declares

$$\rho = 0.85, \qquad c_H = 7.14, \qquad k = 0.015, \qquad r_0 = 1.1, \qquad r_1 = 2.3,$$

Table A.5. Parameters of the example.

Object	Symbol	Value	Kind
Class values	$h,\ \ell$	$10,\ 1$	declared
Reserve	p	$1/2$	declared
Zero-cost probability	ρ	$1/4$	declared
Preparation costs	$c_L,\ c_H$	$0,\ 6$	declared
Trading cost	k	$1/50$	declared
Prior	π	$1/2$	declared
Incumbent strengths	$r_0,\ r_1,\ r_2$	$6/5,\ 3,\ 18/5$	declared
Slot mass ratio	$u = \mathrm{e} - 1$	1.718282	declared
Liquidity order probability	β	0.294655, an exact decimal	declared
Presence probability	$\lambda = u\beta/(1 + u\beta)$	0.336122	derived
Liquidity masses	$b,\ b_0$	0.195615, 0.272648	derived
Posterior bounds	$m(\pi),\ M(\pi)$	0.268941, 0.731059	derived
Research cost and cap	$a,\ \bar{\lambda}$	$1/40,\ 0.336122$	declared
Cost distribution	$\bar{c}_0;\ \varepsilon$	$1/5;\ 1/10,\ 1/1000$	declared
Signal accuracies	$\alpha_I,\ \alpha_C$	$0.94,\ 0.505$	declared
Four-class values	$\theta_1, \ldots, \theta_4$	$1,\ 1001/1000,\ 9999/1000,\ 10$	declared
Four-class prior	π_j	$1/4$ each	declared
High cost, access outside recruitment	c_H	$12/5$	declared
Reserves	p	$101/100,\ 11/10,\ 31/20$	declared

Notes. The upright e is Euler's number. The research cap equals the example's presence probability, so research at the cap reproduces the example's slot. The rows below the rule declare the inputs of single results. The economies of Table A.4 carry their own declarations in that table.

and the accuracies $(\alpha_I, \alpha_C) = (0.70, 0.75)$. Its primitive margins ρ, $1 - \rho$, $r_0 - \ell$, $r_1 - r_0$, $h - r_1$, c_H and k are positive. Its values compare two strengths within the signal example.

The margins table. Table A.6 lists every strict hypothesis of the paper with its margin in the example. Each margin is the exact difference between the two sides of its inequality, signed so that a positive number means that the hypothesis holds; decimals round exact values. A margin that depends on e lies between two rational bounds rounded outward, and its sign is the common sign of both bounds. A negative entry marks a comparison economy built to fail its hypothesis, and the table says which. The bound $\beta \le 1/3$ restricts the domain of the model, and the table records its distance.

The margin of (H3) at the upper end of the band is one distance in three units: $c_H - B_{r_2}(M) = 0.002689$ in cost units, $\tau(r_2) - M = 0.000334$ in posterior units, and $r_2 - r_C = 0.007341$ in strength units. Given the auction primitives it depends only on the top posterior and on the place of r_2, and in cost units it is the smallest margin of Proposition 2.

The parameter set around the example. Each row of Table A.6 whose hypothesis holds in the example, taken with the sign that the row requires, is a strict inequality between continuous functions of the primitives. Together these rows define an open set of primitives that contains the example;

Table A.6. Strict hypotheses and their margins in the example.

Result	Hypothesis	Margin	Value in the example
Proposition 2 and Corollary 1 (prior $\pi = 1/2$)			
(H1) left	$B_0(\pi) < c_H$	$c_H - B_{r_0}(\pi)$	1.195833
(H1) right	$c_H < B_1(M)$	$B_{r_1}(M) - c_H$	0.217155
		$M - \tau(r_1)$	0.026059
(H2) left	$\Delta_0 < k$	$k - \Delta_T(r_0)$	0.003333
(H2) right	$k < \rho\kappa\Delta_1$	$\rho\kappa\Delta_T(r_1) - k$	0.024824
(H3)	$B_2(M) < c_H$	$c_H - B_{r_2}(M)$	0.002689
		$\tau(r_2) - M$	0.000334
Implied	$B_1(\pi) < c_H$	$c_H - B_{r_1}(\pi)$	1.708333
	$k < \rho\kappa\Delta_2$	$\rho\kappa\Delta_T(r_2) - k$	0.043127
	$B_2(\pi) < c_H$	$c_H - B_{r_2}(\pi)$	1.865278
Slot band	$u > u_{\text{lo}}$	$u - 82/59$	0.328451
	$u < u_{\text{hi}}$	$2686/1559 - u$	0.004617
	λ in its band	distance to lower, upper end	0.045583, 0.000599
Prior range	π in its range	distance to lower, upper end	0.032150, 0.000424
Domain	$\beta \le 1/3$	$1/3 - \beta$	0.038678
Ownership	$e_H W_H(F_1) > \rho W_H(F_0)$	$e_H - \rho\,(W_H = 1)$	0.398803
	$b + \lambda > 1/2$ (sufficient form)	$b + \lambda - 1/2$	0.031737
Outbidding	$r_2 < h$	$h - r_2$	32/5
Decisive advantage	$(h^2 - p^2)/(2h) \le c_H$	$c_H - (h^2 - p^2)/(2h)$	81/80
Affordability	$c_H \le h - r_2$	$h - r_2 - c_H$	2/5
Recruitment bound	$h - c_H + \sqrt{(h - c_H)^2 - p^2} < h$	$c_H - \sqrt{(h - c_H)^2 - p^2}$	2.031373
Proposition 3, Corollary 2 and Corollary A.2 (the example's path, Lemmas A.10 and A.11)			
Path	$\Delta_T > 0$, $g_H > g_L$ on the path	$\min \Delta_T$; $\min(g_H - g_L)$	1/60; 1451/180
	$\tau(r_0) > 1/2$	$\tau(r_0) - 1/2$	0.133117
	$J(r_0) < k$	$k - J(r_0)$	1283/92400
	$\rho J(r_1) > k$; $\rho J(r_2) > k$	$\rho J - k$	7/240; 0.043048
	$\tau(r_2) < 1 - \rho$	$1 - \rho - \tau(r_2)$	27/1451
	$\tau(r_N) > 1/2$	$\tau(r_N) - 1/2$	0.154463
	$u < 19/3$	$19/3 - u$	4.615052
Uniform affordability	$c_H < h - r_2$	$h - r_2 - c_H$	2/5
Balance	$\left(\Delta_T\,(g_H - c_H)\right)' \ge 0$	least value of $\left(\Delta_T\,(g_H - c_H)\right)'/\Delta_T'$	9667/8280
Squared balance	$\left(\Delta_T\,(g_H - c_H)^2\right)' > 0$	least value of $\left(\Delta_T\,(g_H - c_H)^2\right)'/[(g_H - c_H)\Delta_T']$	937/5520
Order	$r_0 < r_A$	$r_A - r_0$	0.193958
	$r_A < r_F$	$r_F - r_A$	0.073169
	$r_F < r_N$	$r_N - r_F$	0.280750
	$r_N < r_J$	$r_J - r_N$	0.218220
	$r_J < r_S$	$r_S - r_J$	0.158051
	$r_S < r_1$	$r_1 - r_S$	0.875852
	$\rho^2\Delta_T(r_1) > k$ (gives $r_S < r_1$)	$\rho^2\Delta_T(r_1) - k$	13/600
	$r_1 < r_C$	$r_C - r_1$	0.592659
	$r_C < r_2$	$r_2 - r_C$	0.007341
Priors	$\pi > \pi_A^{\text{lo}}$	$\pi - \pi_A^{\text{lo}}$	0.103825
	$\pi < \pi_A^{\text{hi}}$	$\pi_A^{\text{hi}} - \pi$	0.140739
	$\tau_A < 1 - \rho$	$1 - \rho - \tau_A$	0.109261
	$\tau_A < \pi_{\max}$	$\pi_{\max} - \tau_A$	0.009320
	purchase at r_* and $\pi_{\max}$ [†]	$\Delta_T(r_*)[1 - M(\pi_{\max})] - k$	0.000343

[†] When an indifferent high-cost rival may prepare with any probability.

Table A.6. Strict hypotheses and their margins in the example (continued).

Result	Hypothesis	Margin	Value in the example
Propositions 4 and A.1, Corollary 3 and Lemma A.18 ($a = 1/40$, $\bar{\lambda} = 0.336122$)			
Weak	$B_{r_0}(1/2) < c_H$	$c_H - B_{r_0}(1/2)$	1.195833
	$\Delta_T(r_0) < k$	$k - \Delta_T(r_0)$	0.003333
Intermediate	$\rho\Delta_T(1.65) < 2k$	$2k - \rho\Delta_T$	0.007992
	$\varphi_a(1.55) > 0$	$\varphi_a(1.55)$	0.004629
	$a < a_{\text{up}}$	$a_{\text{up}} - a$	0.011763
	$\lambda_R(1.65) < \bar{\lambda}$	$\bar{\lambda} - \lambda_R$	0.019439
	$a > a_{\text{lo}}(\bar{\lambda})$	$a - a_{\text{lo}}$	0.003531
	$a < a_{\text{mono}}$	$a_{\text{mono}} - a$	0.136713
Cheap research	φ_a^* rises on $[r_0, 7/4]$	lower bound on φ_a'	0.033767
	$\varphi_a^*(r_0) < 0$	$-\varphi_a^*(r_0)$	0.022693
	$\varphi_a^*(r_{\bar{\lambda}}) > 0$	$\varphi_a^*(r_{\bar{\lambda}})$	0.017446
Strong	$\lambda_E(r_1) < \bar{\lambda}$	$\bar{\lambda} - \lambda_E(r_1)$	0.045583
	$\tau(r_1) < M_{\bar{\lambda}}$	$M_{\bar{\lambda}} - \tau(r_1)$	0.026059
	$\bar{\lambda} < \lambda_E(r_2)$	$\lambda_E(r_2) - \bar{\lambda}$	0.000599
	$M_{\bar{\lambda}} < \tau(r_2)$	$\tau(r_2) - M_{\bar{\lambda}}$	0.000334
	(U) at r_1, r_2	$\rho\Delta_T m_{\bar{\lambda}} - k - a\bar{\lambda}$	0.016421, 0.034723
Branch	$r_R < 1.55$	$1.55 - r_R$	0.048977
	$1.65 < r_{\bar{\lambda}}$	$r_{\bar{\lambda}} - 1.65$	0.027633
	$r_{\bar{\lambda}} < r_N$	$r_N - r_{\bar{\lambda}}$	0.070245
Floor margin	φ_ρ rises on $[r_0, r_2]$	smallest piece bound on φ_ρ'	0.011748
	$r_N < r_Q$	$r_Q - r_N$	0.419118
	$r_Q < r_U$	$r_U - r_Q$	0.267050
	$\lambda_E(r_Q) < \bar{\lambda}$	$\bar{\lambda} - \lambda_E(r_Q)$	0.101865
Proposition A.2 (full orders held fixed)			
Weak row	$c_H < B_0(M)$	$B_{r_0}(M) - c_H$	0.879843
Proposition A.3 ($\bar{c}_0 = 1/5$, half-width $\varepsilon = 1/10$)			
Supports	$\bar{c}_0 < B_1(m)$	$B_{r_1}(m) - \bar{c}_0$	2.166179
	$B_0(\pi) < c_-$	$c_- - B_{r_0}(\pi)$	1.095833
	$c_+ < B_1(M)$	$B_{r_1}(M) - c_+$	0.117155
Third economy	$\bar{c}_0 < B_2(m)$	$B_{r_2}(m) - \bar{c}_0$	2.072133
	$B_2(M) < c_-$ (fails by design)	$c_- - B_{r_2}(M)$	−0.097311
	same at $\varepsilon = 1/1000$	$c_H - \varepsilon - B_{r_2}(M)$	0.001689
Proposition A.5 (four classes) and the three-class collision			
Separation	$\rho_{\mathcal{S}} > k$	$\rho_{\mathcal{S}} - k$	0.024718
	weak range below k	k minus the range	0.003333
	weak prior profit below c_H	c_H minus the profit	1.195875
	$\bar{g}_{\mathcal{H}} > \bar{g}_{\mathcal{L}}$	$\bar{g}_{\mathcal{H}} - \bar{g}_{\mathcal{L}}$	8.332667
	top group profit above c_H	profit minus c_H	0.216834
Collision	$\beta > 1/4$	$\beta - 1/4$	0.044655
	$0 < \psi < 1$	ψ; $1 - \psi$	0.940709; 0.059291
	strict class choices	choice gaps	0.260795; 0.020627; 0.024768
Proposition A.6: each value cell gives the signal example, then the example			
Private exclusion	$B_{r_0}(\alpha_C) < c_H$	$c_H - B_{r_0}(\alpha_C)$	0.052500; 1.150917
Joint entry	$c_H < B_{r_1}(\mu_+^C(M^I))$	$B_{r_1}(\mu_+^C(M^I)) - c_H$	0.045265; 0.020731
Weak order	$(2\alpha_I - 1)\bar{\Delta}_P(r_0) < k$	$k - (2\alpha_I - 1)\bar{\Delta}_P(r_0)$	0.001455; 0.002611
Strong order	$k < (2\alpha_I - 1)\rho m\Delta_T(r_1)$	$(2\alpha_I - 1)\rho m\Delta_T(r_1) - k$	0.018594; 0.019445
One pair	$B_{r_1}(\mu_-^C(M^I)) < c_H$	$c_H - B_{r_1}(\mu_-^C(M^I))$	4.159475; 0.048822
Control, the example	no-signal top excluded	$c_H - B_{r_1}(M^I)$	0.013904
	$\alpha_C \geq \alpha_I$ impossible	weak bound minus k	0.049030

Table A.6. Strict hypotheses and their margins in the example (continued).

Result	Hypothesis	Margin	Value in the example
Proposition A.9 and the example's reserves: the trading, prior-cost and top-posterior margins of Appendix A.7			
Low class excluded, points	three margins at r_0		0.038167, 1.512521, 0.561220
	three margins at r_1		0.047239, 1.835008, 0.089706
Low class excluded, bands	three margins at r_0		0.048076, 1.552083, 0.503375
	three margins at r_1		0.047572, 1.850833, 0.066568
Deterrence	three margins at r_0		0.084215, 1.775000, 0.177445
	three margins at r_1 (top-posterior margin negative)		0.057405, 1.950208, −0.078730
	$p_H < p < h - c_H/M$	distance to each end	0.224730, 0.242723
	$p_H > r_0$	$p_H - r_0$	0.125270
Section 8 (full two-sided trade and the two clusters)			
Full trade	$r_S < r_C$; $r_S < h$	$r_C - r_S$; $h - r_S$	1.468510; 7.875852
	$M > 5/8$	$M - 5/8$	0.106059
Band	full trade at r_1 for every share in the band	weakest bound	0.024768
	$v < 200/259$	$200/259 - v$	0.044849
Price gap	$\ell^2 - 3p^2 > 0$	$\ell^2 - 3p^2$	1/4
Winner gap	$D(r) > \Omega(F_r)$ at r_1; at r_2	$D(r) - \Omega(F_r) = \ell/r$	1/3; 5/18
Clusters	$r_F < 1.55$; $1.65 < r_N$	distances	0.082873; 0.097878
	purchase at 1.55; low class abstains at 1.65	$\Delta_T m - k$; $k - \rho\Delta_T/2$	0.006243; 0.003996
	purchase recruits at 1.65	$B(M) - c_H$	0.713202

with the bound $\beta \le 1/3$, the set is relatively open in the model domain. The rows that define the set leave out the third economy of Proposition A.3 at the half-width $1/10$, which is built to fail its hypothesis; the economy with the same means in the last panel, whose negative top-posterior margin is its purpose; the row that shows that $\alpha_C \ge \alpha_I$ is impossible in the example; and exact ties, such as $r = r_C$, $u = u_{\text{lo}}$ or the informed share $1 - 2k/(\rho\Delta_T(r_1))$ at the right end of the band, which are equilibria under the stated tie rules. For Section 8, the section of the set through the example is $0 < \beta < 1/3$ with the slot band and $r_S(M) < r < h$; on it the uniqueness margin $\rho\Delta_T(r)m - k$ is positive at every strength and tends to zero as $r \downarrow r_S$. The other sections through the example are the slot band and the prior range of Corollary 1 (Table 2), the cost window of Corollary 1(e), the cap strip and the cost window of Proposition A.1, the support conditions of Proposition A.3, which allow every half-width below $c_H - B_{r_2}(M) \approx 0.002689$ for all three parts, and the reserve intervals of Appendix A.7.

Table A.6. Strict hypotheses and their margins in the example (continued): the economies of Table A.4.

Economy	Prior cost	Top posterior	Weak trade	Purchase	Sale	Exclusion	Ownership
The example	1.195833	0.217155	0.003333	0.024824	0.024824	0.002689	0.398803
Moderate gap	0.033452	0.030137	0.000810	0.003603	0.003603	—	0.398803
Narrow bands	1.195660	0.217192	0.002986	0.024833	0.024833	0.002658	0.398803
Low prior	1.460455	0.109179	0.005455	0.066983	0.012698	0.058137	0.398803
High prior	0.562273	0.123413	0.005455	0.012698	0.066983	0.132569	0.398803
Wide classes	0.021000	0.058891	0.006667	0.023541	0.127671	0.314551	0.349742
Same means	0.025000	−0.018122	0.020000	0.026343	0.137174	0.447727	0.398803

Notes. Prior cost is $c_H - B_0(\pi)$, the left side of the high-cost window; top posterior is $B_1(M) - c_H$, its right side; weak trade is $k - \Delta_0$, the left side of the trading-cost window; purchase and sale are $\rho(1 - M)\Delta_1 - k$ and $\rho m\Delta_1 - k$, the trading margins of the two correct orders at F_1; exclusion is $c_H - B_2(M)$, the exclusion condition; and ownership is $[\rho + (1 - \rho)(b + \lambda)]W_H(F_1) - \rho W_H(F_0)$. The moderate-gap economy has two strengths. In the last economy the two classes have the same means: its negative top-posterior margin keeps entry at ρ, and its ownership margin uses the recruitment coefficient that the economy does not reach.